PLTL Calculus 2 Spring 2019 Session 2 - Water Clock Design

Background Knowledge

1. **Slice and Sum**
   The idea is to find the volume of a cross-sectional slice of the solid, and then add up all such volumes. We want to use a very thin slice, of thickness dx or dy, since we are going to be taking the limit as the thickness of each slice goes to zero to come up with an integral that gives us the volume of the whole solid. So, all we need to do to get the volume of a slice is multiply the thickness by the surface area of the slice.

   $$\text{Volume of Solid} = \int_a^b (\text{Area of Slice})\,(\text{Thickness})$$

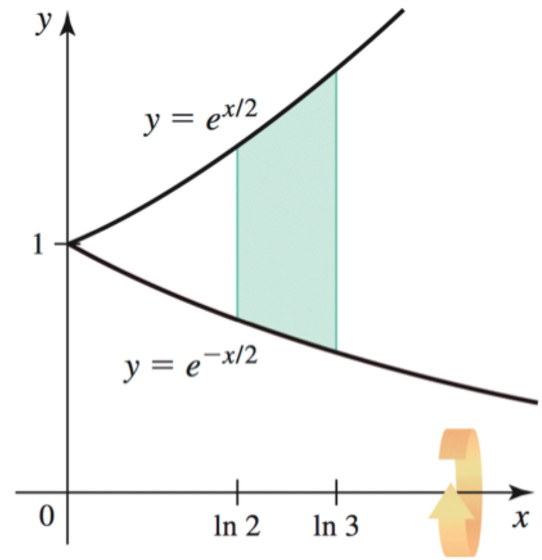
   <u>Example</u>
   A solid has a base that is bounded by the curves $y = x^2$ and $y = 2 - x^2$ in the xy-plane. Cross sections through the solid perpendicular to the base and parallel to the y-axis are semicircular disks. Find the volume of the solid.

2. **Volume of Revolution - Washer method**
   A specific case of the slice and sum method comes when we rotate a region from the xy-plane around an axis, and use slices that are PERPENDICULAR to the axis of revolution. This produces disks, or disks with holes in them that we call washers. Since the face of each slice is a circle, we can find the volume of the resulting solid using this formula:

   $$\text{Volume of Solid} = \pi \int_a^b \left[(\text{outer radius})^2 - (\text{inner radius})^2\right](\text{thickness})$$

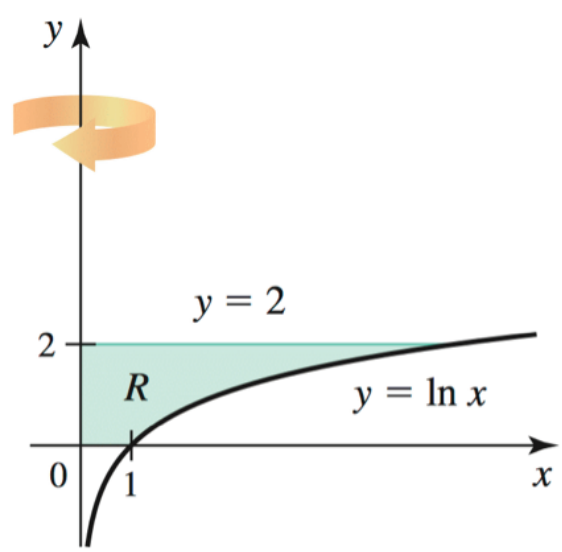
Example

Set up an integral that would give the volume of the region generated by revolving the region R about the x-axis.



Example

Set up an integral that would give the volume of the region generated by revolving the region R about the y-axis.

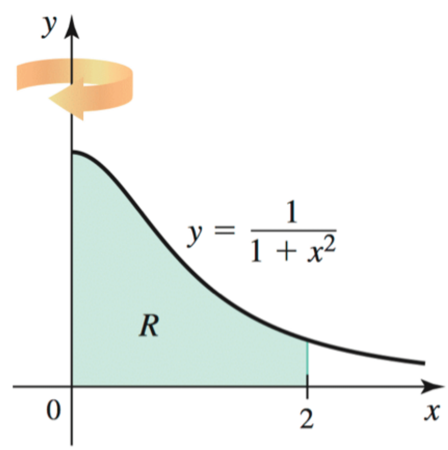


3. **Volume of Revolution - Shell Method**
Another specific case of the slice and sum method comes when we rotate a region from the xy-plane around an axis, and use slices that are PARALLEL to the axis of revolution. This produces cylindrical shells, and we can find the volume of the resulting solid using this formula:

$$\text{Volume of Solid} = 2\pi \int_a^b (\text{radius of cylinder})\,(\text{height of cylinder})\,(\text{thickness})$$

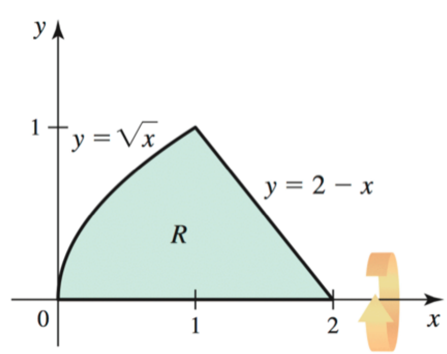
Example

Set up an integral that would give the volume of the region generated by revolving the region R about the y-axis.



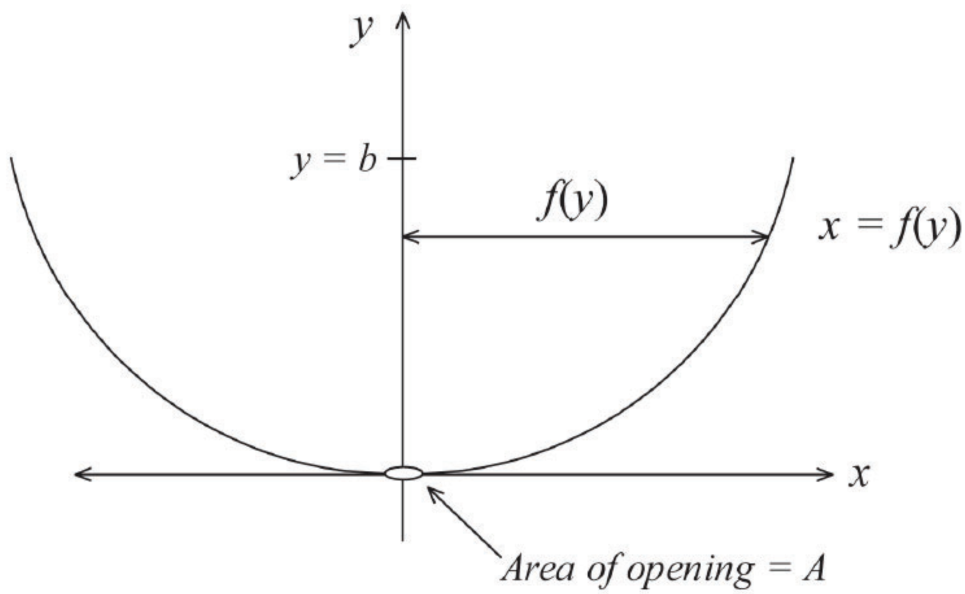
Example

Set up an integral that would give the volume of the region generated by revolving the region R about the x-axis.



Application

Before mechanical and digital clocks were invented, time was kept by water clocks (and other devices). A water clock consists of a large container with a hole in the bottom. When filled with water, the water drains out through the hole in such a way that the surface of the water drops at a constant rate. In this way, the clock is easily calibrated: lines are marked on the container with equal spacing, and the distance between two lines always corresponds to the same interval of time (like an hour).



The goal would be to find the function formula f that would define the shape of exactly such a water clock.

1. Assume that the water in the clock forms a solid of revolution generated when the curve $x = f(y)$ between $y = 0$ and $y = b$ is revolved around the y-axis. Suppose that the water level is at a variable point y. Use the disk method to find an integral that would give the volume V of water in the container - use t as the dummy variable for integration.

2. As the water level in the clock drops, both the volume of water V and the water level y change over time. Use the Fundamental Theorem of Calculus to differentiate both sides of the volume formula with respect to time t.

3. Explain why dy/dt is a constant. Since we are assuming dy/dt will be given (-1 inch per hour, for example), we need to know something about dV/dt. **Torricelli's Law** says that the rate of change of the volume of water in a container open to the air draining through a hole with area A is

$$\frac{dV}{dt} = -A\sqrt{2gy}$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity and y is the depth of the water in the container. As an example, let $A = 0.02$ m^2 be the area of the hole, and graph dV/dt as a function of y on the window $0 \leq y \leq 1$, $-0.1 \leq dV/dt \leq 0$. Describe how the draining rate varies with the depth of the water.

4. Now assume $dy/dt = -k$ m/s is the constant rate at which the water level drops, where $k > 0$. Use Torricelli's Law and Step 2 to show that the function f that gives the shape of the water clock is

$$f(y) = \sqrt[4]{\frac{2gA^2y}{\pi^2k^2}}$$

5. Suppose you want to design a 24-hour clock that is $b = 1$ m high. This means that the water level must drop 1/24 m each hour, so $k = 1/24$ m/hr. Suppose the hole in the bottom of the tank has an area of $A = 0.02$ m^2, so that the radius is approximately 8 cm or 3 in. Find and graph the function that describes the container; find the upper radius of such a container; and find the volume of water needed to fill this container.