PLTL Calculus 1 Fall 2019 Session 10 – Planets

Background Knowledge

1. In class last week you talked about integration by substitution, where you use a change of variables to make an integration problem easier.

   Example Use substitution to calculate the value of the definite integral

$$\int_0^8 \frac{3}{(2 - x/4)^2}\,dx$$

2. Another topic involving integration is the idea of the *average value* of a function $y = f(x)$ over an interval $[a, b]$. The average value $\bar{f}$ was given by this formula:

$$\bar{f} = \int_a^b f(x)dx$$

Related to this is the *Mean Value Theorem*, which states that a continuous function $f(x)$ over an interval $[a, b]$ must attain its average value at least once on the interval. You can find the location(s) of these points by solving the equation $f(x) = \bar{f}$ and keeping only the solutions that live in the interval $[a, b]$.

Example

Find the average value of the upper part of the ellipse $4x^2 + y^2 = 1$ on the interval $\left[0, {}^{1}\!/_{2}\right]$, and find the points on the ellipse where that average value is achieved.

3. Another topic covered in class was recovering quantities from their rate of change. Given the rate of change $Q'(t)$ of a quantity $Q(t)$, We know the following:

   A. The *net change* in Q between $t = a$ and $t = b$ is
   $$Q(b) - Q(a) = \int_a^b Q'(t)dt$$
   B. Given the initial value $Q(a)$, the future value of Q at time $t > a$ is
   $$Q(t) = Q(a) + \int_a^t Q'(t)dt$$

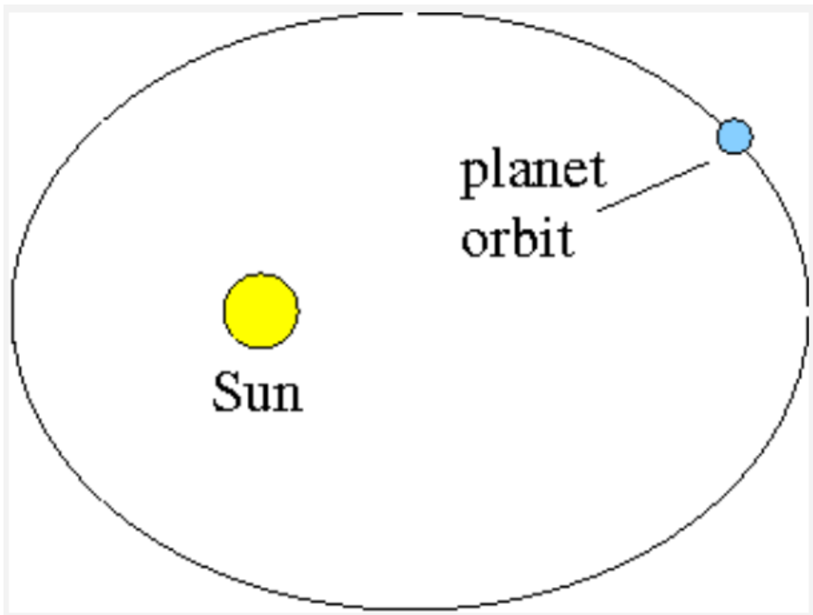
This last equation can be interpreted as: future value = initial value + net change.

Example

Use the Fundamental Theorem of Calculus to find the position and velocity at time $t = 4$ of an object moving along a straight line with the following characteristics:
$$a(t) = \frac{20}{(t+2)^2}; \;\; v(0) = 20; \;\; s(0) = 10$$

Planetary orbits

The orbits of planets around the Sun in our solar system are not circular with the Sun at the center. In reality, the orbits are elliptical, and the Sun lives at one of the two focus points of the ellipse, as in this picture:



Let $(0,0)$ be the center of the ellipse, let a be the major radius, and let b be the minor radius. The equation for this ellipse is then given by:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

<u>DO THIS</u>:

1. Let d^2 denote the square of the distance from a planet to the center of the ellipse. Integrate over the interval $[-a,a]$ to find the average value of this distance, and find the coordinates of the planet when it is exactly this distance from the center.

2. If you assume that $0 < b < a$, the coordinates of the Sun (at one focus point) would be either $\left(-\sqrt{a^2 - b^2}, 0\right)$ as shown in the picture, or $\left(\sqrt{a^2 - b^2}, 0\right)$, which may be easier to use for calculations. Now, let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a, a]$ to find the average value of this distance, and find the coordinates of the planet when it is exactly this distance from the Sun.

Variable gravity

At Earth's surface, the acceleration due to gravity is approximately $g = 9.8\ m/s^2$ (with local variations). However, the acceleration decreases with distance from the surface according to Newton's Law of Gravitation; at a distance of y meters from Earth's surface, the acceleration is actually given by

$$a(y) = -\frac{g}{(1 + y/R)^2}$$

where $R = 6.4 \times 10^6\ m$ is the radius of the Earth.

For this problem, we suppose that a projectile is launched upward with an initial velocity of $v_0\ m/s$. Let $v(t)$ be the velocity and $y(t)$ the height in meters above the surface t seconds after launch. Neglecting forces such as air resistance, we get that the equation of motion for the projectile is given by

$$\frac{1}{2}\frac{d}{dy}(v^2) = a(y).$$

<u>DO THIS</u>:

1. Explain why ${}^{dv}/_{dt} = a(y)$ and ${}^{dy}/_{dt} = v(t)$. Then use the chain rule to show that

$$\frac{dv}{dt} = \frac{1}{2}\frac{d}{dy}(v^2).$$

2. Integrate both sides of this equation with respect to y, using the fact that when $y = 0$, $v = v_0$, to show that

$$\frac{1}{2}\left(v^2 - {v_0}^2\right) = gR\left(\frac{1}{1 + y/R} - 1\right).$$

3. Use the fact that $v = 0$ at the maximum height of the projectile to determine that the maximum height is

$$y_{max} = \frac{R{v_0}^2}{2gR - {v_0}^2}$$

and show that the value of v_0 needed to put the projectile into orbit (the escape velocity) is

$$\sqrt{2gR}.$$