PLTL Calculus 2 Session 1 - Water Clock Design

PLTL students should attempt ALL the highlighted problems in this activity BEFORE joining the peer-led session. Students will be asked during the session to provide their answers. If you get stuck on any part the peer leader can help you.

Background Knowledge

1. **Slice and Sum**
   The idea is to find the volume of a cross-sectional slice of the solid, and then add up all such volumes. We want to use a very thin slice, of thickness dx or dy, since we are going to be taking the limit as the thickness of each slice goes to zero to come up with an integral that gives us the volume of the whole solid. So, all we need to do to get the volume of a slice is multiply the thickness by the surface area of the slice.

$$\text{Volume of Solid} = \int_a^b (\text{Area of Slice})\,(\text{Thickness})$$

2. **Volume of Revolution - Washer method**
   A specific case of the slice and sum method comes when we rotate a region from the xy-plane around an axis, and use slices that are PERPENDICULAR to the axis of revolution. This produces disks, or disks with holes in them that we call washers. Since the face of each slice is a circle, we can find the volume of the resulting solid using this formula:

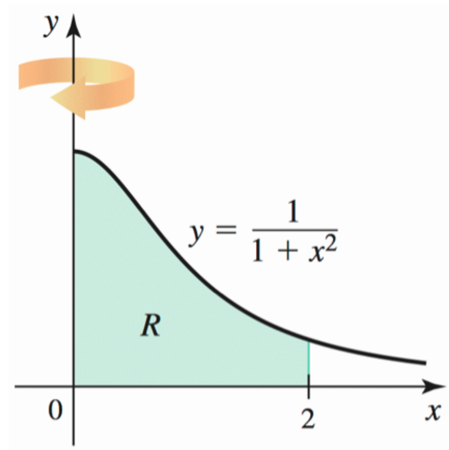
$$\text{Volume of Solid} = \pi \int_a^b \left[(\text{outer radius})^2 - (\text{inner radius})^2\right](\text{thickness})$$

3. **Volume of Revolution - Shell Method**
   Another specific case of the slice and sum method comes when we rotate a region from the xy-plane around an axis, and use slices that are PARALLEL to the axis of revolution. This produces cylindrical shells, and we can find the volume of the resulting solid using this formula:

$$\text{Volume of Solid} = 2\pi \int_a^b (\text{radius of cylinder})\,(\text{height of cylinder})\,(\text{thickness})$$

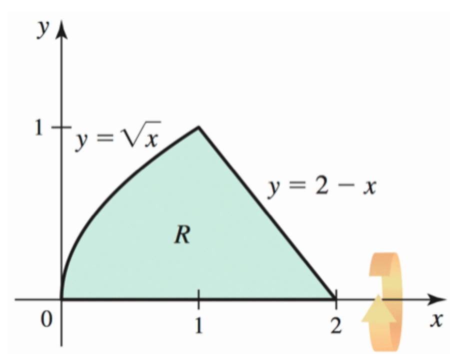
Example

Set up an integral that would give the volume of the region generated by revolving the region R about the y-axis.



Example

Set up an integral that would give the volume of the region generated by revolving the region R about the x-axis.



4. The **arc length** of a curve of the form $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

5. The **surface area** generated by revolving the curve $f(x)$ from $x = a$ to $x = b$ around the x-axis is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

Example

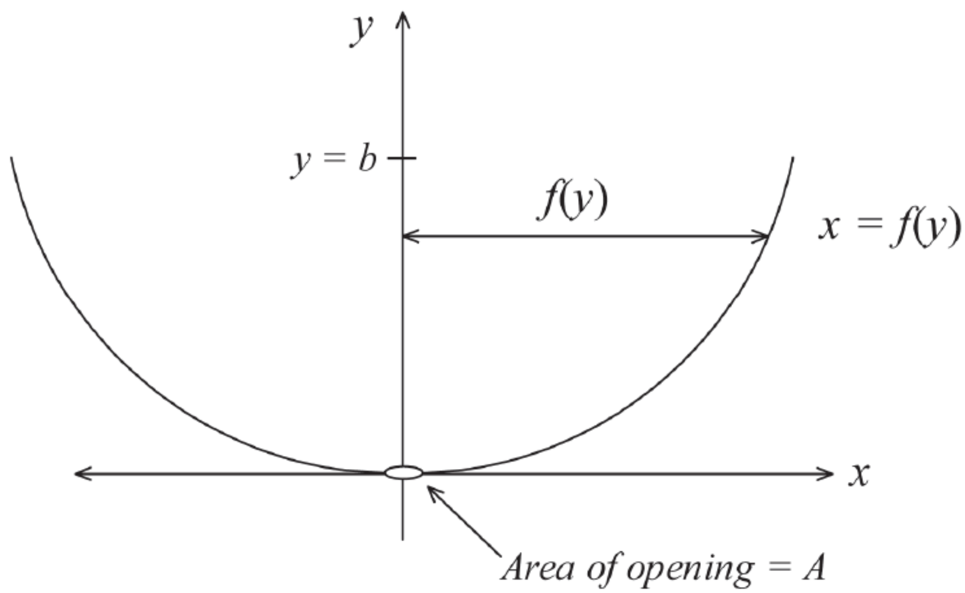
Let $f(x) = \dfrac{x^4}{2} + \dfrac{1}{16x^2}$, and let R be the region bounded by the graph of f and the x-axis over the interval $[0, 2]$.

(a) Set up an integral that would give the length of the curve $y = f(x)$ on $[1, 2]$.

(b) Set up an integral that would give the area of the surface generated when the graph of f on $[1, 2]$ is revolved around the x-axis.

Application

Before mechanical and digital clocks were invented, time was kept by water clocks (and other devices). A water clock consists of a large container with a hole in the bottom. When filled with water, the water drains out through the hole in such a way that the surface of the water drops at a constant rate. In this way, the clock is easily calibrated: lines are marked on the container with equal spacing, and the distance between two lines always corresponds to the same interval of time (like an hour).



The goal would be to find the function formula f that would define the shape of exactly such a water clock.

1. Assume that the water in the clock forms a solid of revolution generated when the curve $x = f(y)$ between $y = 0$ and $y = b$ is revolved around the y-axis. Suppose that the water level is at a variable point y. Use the disk method to find an integral that would give the volume V of water in the container - use t as the dummy variable for integration.

2. As the water level in the clock drops, both the volume of water V and the water level y change over time. Use the Fundamental Theorem of Calculus to differentiate both sides of the volume formula with respect to time t.

3. Explain why dy/dt is a constant. Since we are assuming dy/dt will be given (-1 inch per hour, for example), we need to know something about dV/dt. **Torricelli's Law** says that the rate of change of the volume of water in a container open to the air draining through a hole with area A is

$$\frac{dV}{dt} = -A\sqrt{2gy}$$

where $g = 9.8\ \text{m/s}^2$ is the acceleration due to gravity and y is the depth of the water in the container. As an example, let $A = 0.02\ \text{m}^2$ be the area of the hole, and graph dV/dt as a function of y on the window $0 \leq y \leq 1$, $-0.1 \leq dV/dt \leq 0$. Describe how the draining rate varies with the depth of the water.

4. Now assume $dy/dt = -k$ m/s is the constant rate at which the water level drops, where $k > 0$. Use Torricelli's Law and Step 2 to show that the function f that gives the shape of the water clock is

$$f(y) = \sqrt[4]{\frac{2gA^2y}{\pi^2k^2}}$$

5. Suppose you want to design a 24-hour clock that is $b = 1$ m high. This means that the water level must drop 1/24 m each hour, so $k = 1/24$ m/hr. Suppose the hole in the bottom of the tank has an area of $A = 0.02$ m^2, so that the radius is approximately 8 cm or 3 in. Find and graph the function that describes the container; find the upper radius of such a container; and find the volume of water needed to fill this container.

PLTL Calculus 2 Session 2 - Integration Methods

In this activity we will go over the integration methods you covered in class over the last week, and work on some applications of these methods.

1. **Trig Substitution**
   Some integrals that have expressions with one of the forms a^2-x^2, a^2+x^2, or x^2-a^2 present may require using trig substitution. It is important to know which trig substitution to use in each case, AND what the corresponding interval for θ should be. Here's a summary:

The Integral Contains . . .	**Corresponding Substitution**	**Useful Identity**
$a^2 - x^2$	$x = a\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for $\|x\| \le a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2}, \text{ for } x \ge a \\ \frac{\pi}{2} < \theta \le \pi, \text{ for } x \le -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

   <u>Example 1</u> Evaluate the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{(16-x^2)^3}}$

Example 2

A total charge of Q is distributed uniformly on a line segment of length $2L$ along the y-axis, centered at the origin. The x-component of the electrical field at a point $(a, 0)$ on the x-axis is given by

$$E_x(a) = \frac{kQa}{2L} \int_{-L}^{L} \frac{dy}{(a^2 + y^2)^{3/2}}$$

where k is a physical constant and $a > 0$.

(a) Use an appropriate trig substitution to evaluate the integral, and confirm that

$$E_x(a) = \frac{kQ}{a\sqrt{a^2 + L^2}}$$

(b) Letting $\rho = Q/2L$ be the charge density on the line segment, show that if $L \to \infty$, then $E_x(a) = 2k\rho/a$.

2. **Integration by Parts**

The formula for integration by parts is

$$\int u\,dv = uv - \int v\,du$$

You split the original integrand into a product, call one factor u and the rest dv, and then integrate dv to get v and differentiate u to get du. When you use these to write out the expression on the right hand side of the formula, you should get an integral that is easier to do than the one you started with.

Consider the set of integrals defined for every nonnegative integer n by

$$I_n = \int x^n e^{-x^2}\,dx.$$

(a) Write down I_1 and evaluate the integral. Can you calculate I_2?

(b) Write down I_3 and evaluate the integral. You will have to use integration by parts and then use your answer to part (a).

(c) Use the result of part (b) to evaluate I_5.

(d) Guess a formula for I_n where n is odd, and confirm your guess by evaluating I_7.

PLTL Calculus 2 Session 3 - More Integration Methods

PLTL students should attempt ALL the highlighted problems in this activity BEFORE joining the peer-led session. Students will be asked during the session to provide their answers. If you get stuck on any part the peer leader can help you.

In this activity we will go over some of the integration methods you covered in class over the last week, and work on some applications of these methods.

1. **Partial Fraction Decomposition**
   When you are trying to integrate a rational function with denominator degree greater than numerator degree, there are many different strategies for using partial fractions, depending on whether the factors in the denominator are linear, quadratic, and or repeated. You can check your class notes or the section in the e-book for a summary of these different methods.

   Example 1 Consider the indefinite integral $\int \frac{x^2+12x-4}{x(x+2)(x-2)}\,dx$.

   (a) Write a partial fraction decomposition for the integrand function in the form

   $$\frac{A}{x}+\frac{B}{x+2}+\frac{D}{x-2} \tag{1}$$

   (b) Integrate the formula given in equation (1) without finding the values of A, B, and D (your peer leader will help you figure out the values of A, B, and D to do part (a) during the live session)

Example 2

A computer algebra system (CAS) was used to determine that the value of the integral

$$\int \frac{2x^4 + 14x^3 + 14x^2 + 51x + 19}{x^5 + x^4 + 3x^3 + 3x^2 - 4x - 4}\, dx$$

was equal to the following antiderivative family:

$$5\ln(x-1) - 3\ln(x+1) - \frac{3}{x+1} + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Use this fact to find the partial fraction decomposition of the function in the integrand.

2. **General Integration Methods**
Cars A, B, and C start from rest and move along a road with the following velocity functions:

$$v_A = \frac{24t}{t+1} \qquad v_B = \frac{24t^2}{(t+1)^2} \qquad v_C = \frac{24t^2}{t^2+1}$$

Use long division to rewrite each velocity function.

Find the position functions for all three cars, $s_A(t), s_B(t), s_C(t)$. Assume that the common starting point is the origin (this will help you determine the value of the integration constant).

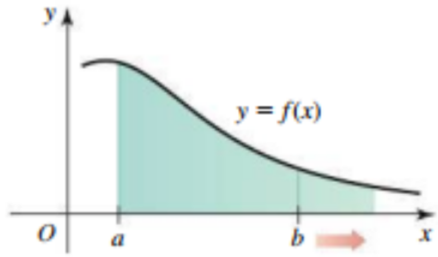
Which car is in the lead after 1 second? After 5 seconds?

3. **Improper Integrals**

You need to be able to deal with integrals where either the interval you are integrating over has infinite length, OR where the function you are integrating has a vertical asymptote within the interval you are integrating over. The Fundamental Theorem of Calculus only works for finite closed intervals where the function you are integrating is continuous, so we have to be careful and use a limit if that is not the case.

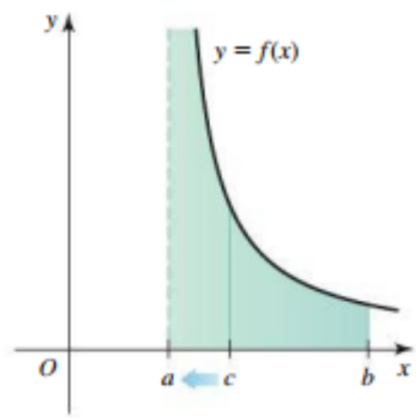
If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$



Suppose f is continuous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$



Example 3 We want to evaluate the integral $\int_{-1}^{1} \frac{1}{x}\,dx$. Explain why the following is incorrect, and then calculate the integral correctly:

$$\int_{-1}^{1} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{1} = \ln(1) - \ln(1) = 0$$

Example 4

The probability that a computer chip fails after a hours of operation is given by

$$0.00005 \int_a^{\infty} e^{-0.00005t}\, dt$$

(a) Find the probability that a computer chip fails after 15,000 hours of operation.

(b) Of the chips still operating after 15,000 hours, what fraction of them will operate for at least another 15,000 hours?

(c) The average lifetime until a computer chip fails is given by the integral below. Calculate this value.

$$0.00005 \int_a^{\infty} te^{-0.00005t}\, dt$$

PLTL Calculus 2 - Session 4

Students should read through the activity and complete the highlighted portions of this activity before the live session with the peer leader on Sunday.

1. **Sequence and Series overview**

   A **sequence** is an ordered list of terms $\{a_1, a_2, a_3, \ldots\}$. Each number in the sequence is called a **term** of the sequence, and the position of each term is called the **index**, represented by the subscript.
   If the terms of a sequence approach a unique number L as n increases, then we say that the sequence **converges** to L, and we write

   $$\lim_{n\to\infty} a_n = L.$$

   Otherwise, we say that the sequence **diverges**.

   For any sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$, the associated **infinte series** is the sum of its terms,

   $$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

   To determine if this adds up to a finite number, form the **sequence of partial sums** $\{S_n\}$ associated with the series:

   $$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= \sum_{k=1}^{n} a_k \end{aligned}$$

   If this sequence $\{S_n\}$ has a limit L, then we say that the infinite series converges to that limit, and we write

   $$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L$$

   Otherwise, we say that the series diverges.

Example 1 Consider the sequence $\{a_n\} = \left\{ \frac{1}{n(n+1)} \right\}$.

(a) Find the first 8 terms of the sequence. Does the sequence appear to have a limit?

(b) Now consider the infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. Find the first 5 terms of the sequence of partial sums $\{S_n\}$. Find a formula for the general term S_n, and make a conjecture about the value of the series.

Example 2

A ball is thrown upward to a height of $h_0 = 20$ meters. After each bounce, the ball rebounds to a fraction $r = 1/2$ of its previous height. Let h_n be the height of the ball after the nth bounce, and let S_n be the total distance the ball has traveled at the moment of the nth bounce.

(a) Find the first 5 terms of the sequence of heights $\{h_n\}$.

(b) Find an explicit formula for the nth term of the sequence $\{h_n\}$. Make a conjecture as to the limit of this sequence.

(c) Find the first 10 terms of the sequence $\{S_n\}$. Make a conjecture as to the total distance traveled by the ball.

2. **Recurrence relations**

Some sequences have terms that are defined using the terms that come before them in the sequence. This kind of relationship could be based on the previous term only, like $a_{n+1} = 6a_n - 4$, or on more than one of the previous terms in the sequence, like $a_{n+1} = 3a_n - 2a_{n-1} + 4a_{n-2}$. The following examples involve working with recurrence relations.

Example 3
A fishery manager knows that her fish population naturally increases at a rate of 1.5% per month, while 80 fish are harvested each month. Let F_n be the fish population after the nth month, where $F_0 = 4000$ fish.

(a) Write out the first 5 terms of the sequence $\{F_n\}$, and try to guess a recurrence relation that generates the sequence.

(b) Does the fish population increase or decrease in the long run? What if the initial population was $F_0 = 5500$ fish instead?

(c) Determine the initial fish population F_0 below which the population decreases in the long run.

Example 4

After many nights of observation, you notice that if you oversleep one night, you tend to undersleep the following night, and vice versa. This pattern of compensation is describes by the following relationship, for $n = 1, 2, 3, \ldots$:

$$x_{n+1} = \frac{1}{2}\left(x_n + x_{n-1}\right)$$

where x_n is the number of hours of sleep you get on the nth night, and $x_0 = 7$ and $x_1 = 6$ are the number of hours of sleep on the first two nights.

(a) Write out the first 7 terms of the sequence $\{x_n\}$ and confirm that the terms alternately increase and decrease.

(b) The terms of the sequence can also be generated by the explicit formula

$$x_n = \frac{19}{3} + \frac{2}{3}b^n$$

What is the value of b?

PLTL Calculus 2 - Session 5 - Economic Stimulus

Suppose that one day a rich relative gives you $\$10,000$ to use however you want, no strings attached. The average American in 2020 would end up saving less than 5% of the $\$10,000$, with the rest of it being spent and going back into the economy. This spending generates revenue through taxes, and it gets spent again. In this way, your money is actually spent many times and it has the effect of stimulating the economy through what is known as the **multiplier effect**. Governments occasionally provide an economic stimulus by offering tax rebates or refunds. In this project we will take a look at a simplified explanation of how the multiplier effect could lead to economic growth, under one particular theory of economics. (It is important to note that not all economists agree that government stimulus spending would have this kind of effect.)

1. Let's go back to the first month after your $\$10,000$ gift, and assume that you save 5% of it, spending 95% of it back into the economy. In this first month, how much do you save and how much do you return to the economy?

2. Now suppose that each month the same cycles repeats: 95% of the current amount in the economy is recirculated. Make a table showing the amount of money that is returned to the economy in months 1, 2, 3, and 4. It is important to keep the formula for how you calculated each amount, as well as the final totals.

3. The important question concerns the cumulative long-term effect of the recirculation of the money after several months. Define **the economic stimulus after n months** to be the total amount of money that is returned to the economy after n months - which is different than the amount of money returned to the economy in month n. For the \$10,000 gift, write a geometric sum for the economic impact after n months, and call it $I(n)$.

4. Using what you know about geometric sums with n terms, write down a formula for $I(n)$ that does not have a summation in it.

5. For the \$10,000 gift, make another table showing the economic impact after n months for $n = 1, 2, 3, 4, 5$.

6. In theory, the recirculation of money continues indefinitely, so we can look at the economic impact after n months as n becomes arbitrarily large. This gives us the **total economic impact** of the original $\$10,000$ gift as the value of a geometric series. Write down that geometric series, explain why it converges, and calculate the value I.

7. Now let's generalize the situation and assume that we have an economic stimulus in the amount of $\$A$ billion dollars. Let r be the spending rate; in the previous example, we used $r = 0.95$. Explain why $0 \leq r \leq 1$, write down a geometric sum for the economic impact after n months in this case, and find the value of the total economic impact I. What would be the total economic stimulus of $350 billion with an 80% spending rate?

8. The total economic stimulus I you just calculated in general is a function of both the initial stimulus amount A as well as the spending rate r. Assume now that we let A be a fixed value, so that $I = I(r) = \dfrac{A}{1-r}$ is a function of just r. What are the domain and range of the total economic impact function in the context of this problem? Computing the derivatives $I'(r)$ and $I''(r)$, is the total economic impact function increasing or decreasing on its domain? Is it concave up or concave down?

9. The sensitivity of the economic impact to changes in (or errors in estimating) the spending rate is important. Suppose the spending rate changes from r to $r + \Delta r$. Use a linear approximation to find the resulting approximate change ΔI in the total economic impact (in terms of A and r).

10. Suppose that with a 350 billion stimulus package that the spending rate increases from $r = 0.3$ to $r = 0.35$. What is the approximate change in the total economic impact?

11. Suppose that with a 350 billion stimulus package that the spending rate increases from $r = 0.8$ to $r = 0.85$. What is the approximate change in the total economic impact?

12. Note that the change in the spending rate is the same ($\Delta r = 0.05$) in both of the previous two calculations. Is the change in the total economic impact greater when r is small or large?

PLTL Calculus 2 Session 6 - Conditional Convergence

Before the live PLTL session on Sunday, students should read the background material for each section and attempt the problems that are highlighted.

Background

In this project we will need to be familiar with series comparison tests and with conditional and absolute convergence. Let's review the concepts and do a few examples.

Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

1. If $a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.

2. If $b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Determine whether $\displaystyle\sum_{k=3}^{\infty} \frac{5}{k^4+10}$ converges or diverges.

Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L\,.$$

1. If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.

3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Determine whether $\displaystyle\sum_{k=20}^{\infty} \frac{5k}{k^4-10}$ converges or diverges.

Absolute and Conditional Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**.

If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

Determine if the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

$$\sum_{k=2}^{\infty} \frac{(-1)^{k+1}k}{\sqrt{k^5}}$$

Application

Please read through this introduction carefully so that you are familiar with the application setting to be discussed!

In its simplest form, a crystal is a regular three-dimensional arrangement of oppositely-charged ions of two elements, for example sodium and chlorine atoms in a salt crystal. Such an arrangement is called a **lattice**. We will work with one- and two-dimensional rectangular lattices, like those in Figure 1:

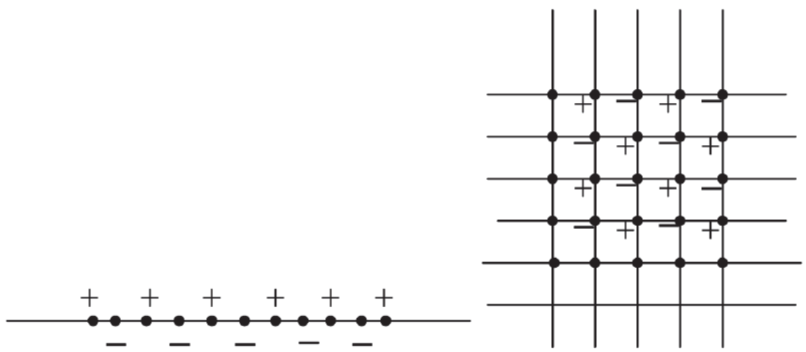


Figure 1

The structure of a crystal lattice is maintained by the balance of forces between the positive and negative ions in the lattice. We want to know the total amount of energy required by the entire lattice to hold a single ion in place - to answer this question, we need to consider the forces between a fixed ion at the origin and all other ions in the lattice.

Two oppositely-charged ions attract each other and two like-charged ions repel each other. The energy associated with a pair of ions is proportional to $\pm 1/r$, where r is the distance between the ions. Choose (-) if the ions have opposite charges and we choose (+) if the ions have like charges. So, the energy required to keep a single ion in place at the origin is proportional to

$$M = \sum_k (\pm)_k \frac{1}{r_k}$$

This series that determines the energy of the lattice is called the **Madelung constant** for the lattice. The sum includes all ions in the lattice except the fixed ion at the origin, and r_k is the distance between the fixed ion and the kth ion. The notation $(\pm)_k$ means to choose either $+$ or $-$ depending on the sign of the kth ion.

1. Consider the one-dimensional lattice in Figure 1 with the fixed ion at the origin corresponding to $k = 0$. To keep things simple, let's assume that the distance between adjacent ions is 1 unit. Explain why the energy associated with this lattice is:

$$M = \sum_{k=1}^{\infty} \frac{(-1)^k}{|k|} + \sum_{k=-\infty}^{-1} \frac{(-1)^k}{|k|} = -2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)$$

2. The series in parentheses in Step 1 is the **alternating harmonic series**, which converges to the number $\ln(2)$. The alternating harmonic series is a **conditionally convergent series**, because the associated series of positive terms $\sum 1/k$ diverges (the harmonic series).

   FACT: by rearranging its terms, a conditionally convergent series can be made to converge to any real number! For example:

   (a) Start with $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$. Multiply both sides of the equation by $1/2$ to get a series for $\frac{1}{2}\ln(2)$.

   (b) Add the two series together to show that

   $$\frac{3}{2}\ln(2) = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$$

   (c) Finally, show that the right side of this equation is just a rearrangement of the original alternating harmonic series. So, this brings out the question: what is the"correct" order in which to evaluate the series for the Madelung constant?

3. Now consider the two-dimensional lattice in Figure 1. Assume again that the distance between any ion and its nearest neighbor is 1 unit, which means that the ions have coordinates (j, k), where j and k are integers and the fixed ion is at $(0, 0)$. Explain why the series for the Madelung constant is

$$M_2 = \sum_{j,k=-\infty}^{\infty} \frac{(-1)^{j+k}}{\sqrt{j^2 + k^2}},$$

with the term for $(j, k) = (0, 0)$ omitted from the series.

4. Write out the eight terms of this series for the ions closest to the origin. How did you choose the order of summation? Are there other possible orders?

5. Use the fact that the series $\sum_{k=0}^{\infty} \frac{1}{k}$ diverges to show that the associated series to M_2 with only positive terms $\sum_{j,k=-\infty}^{\infty} \frac{1}{\sqrt{j^2+k^2}}$ diverges. It is known that the series M_2 actually converges, which means that the series for M_2 is conditionally convergent. So the same problem with rearrangements that we saw in the one-dimensional case comes up here also.

6. Now we can work on the actually physical problem we are interested in, the three-dimensional case.

   (a) Make a rough sketch of a few ions in a regular three-dimensional lattice in which the distance between nearest neighbor ions is 1 unit.

   (b) Omitting the term with $j = k = m = 0$, explain why the Madelung constant in this case is

$$M_3 = \sum_{j,k,m=-\infty}^{\infty} \frac{(-1)^{j+k+m}}{\sqrt{j^2 + k^2 + m^2}}$$

(c) Show that M_3 is a conditionally convergent series.

So we have well-defined physical constants M_1, M_2, and M_3 specified in terms of a conditionally convergent series, which has an infinite number of possible values depending on the ordering of the terms. It turns out that for M_2 the order of summation should start near the origin and work outward along squares, and for M_3 the order of summation should start near the origin and work outward along cubes.

Conclusion: order matters with conditional convergence!

PLTL Calc 2 Session 7 - Series Approximations to π

Mathematicians have been approximating π for several millennia. For example, the Babylonians took π to be equal to $3\frac{1}{8}$, and the Egyptians used the value $\left(\frac{4}{3}\right)^4$.

In the third century BCE Archimedes concluded that $3\frac{10}{71} < \pi < 3\frac{1}{7}$, and the Chinese matematician Liu Hui produced an estimate equivalent to $\pi \approx 3.141024$ in the third century CE.

Today, approximations that are correct to over one quadrillion digits have been calculated using ingenious methods and powerful computers. In this project, we will explore some of the most common approximations which rely on infinite series.

1. An approximation attributed to both James Gregory (1683-1675) and Gottfried Leibniz (1646-1716) relies on the Taylor series for the inverse tangent function, $f(x) = \tan^{-1}(x)$.

   (a) Use the Taylor series for $1/(1+x^2)$ and integration to find a formula for the Taylor series for $f(x) = \tan^{-1}(x)$, and determine that this series converges as long as $-1 \le x \le 1$.

   (b) Substitute $x = 1$ into the Taylor series, which gives you a series that converges to $\tan^{-1}(1) = \pi/4$.

(c) Write the first ten terms of the sequence of partial sums for the series. Assuming that a calculator gives the “exact” value of π, what is the error in the approximation to $\pi/4$ using ten terms?

(d) Use the Remainder in Alternating Series Theorem to estimate the error introduced when the series is terminated after 100 terms.

(e) How many terms of this series must be used to obtain an approximation to π with an error no greater than 10^{-6}?

2. The Gregory/Leibniz series converges very slowly; the reason for this is because with $x = 1$, the powers of x in the Taylor series do not decrease in size. Here is an idea for obtaining better approximations.

(a) There is an addition rule for tangent that says

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}.$$

Let $p = \tan(x)$ and $q = \tan(y)$. Show that $\tan^{-1}(p) + \tan^{-1}(q) = \tan^{-1}\left(\frac{p+q}{1-pq}\right)$.

(b) The objective is to find small numbers p and q so that $\frac{p+q}{1-pq} = 1$. If it can be done, then we would have $\tan^{-1}(p) + \tan^{-1}(q) = \pi/4$, where a series can be used to approximate $\tan^{-1}(p)$ and $\tan^{-1}(q)$.

Show that with $p = 1/2$ and $q = 1/3$, we have

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right).$$

(c) Write the series for $\tan^{-1}(1/2)$ and $\tan^{-1}(1/3)$. Again assuming that a calculator gives the "exact" value of π, what is the error in the approximation to $\pi/4$ using ten terms in this case?

(d) Use the Remainder in Alternating Series Theorem to estimate the error introduced when the series are terminated after 100 terms.

(e) How many terms of the series must be used to obtain an approximation to π with an error no greater than 10^{-6}? Compare this approximation to the previous one for speed of convergence.

3. The idea used in part 2 was extended by John Machin in 1706 to obtain an improved method for approximation π. Machin's formula is

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

(a) Write the series for $\tan^{-1}(1/5)$ and $\tan^{-1}(1/239)$. Again assuming that a calculator gives the "exact" value of π, what is the error in the approximation to $\pi/4$ using ten terms in this case?

(b) Use the Remainder in Alternating Series Theorem to estimate the error introduced when the series are terminated after 100 terms.

(c) How many terms of the series must be used to obtain an approximation to π with an error no greater than 10^{-6}? Compare this approximation to the previous ones for speed of convergence, and figure out how many terms of the series were needed to approximate π to 100 decimal places.

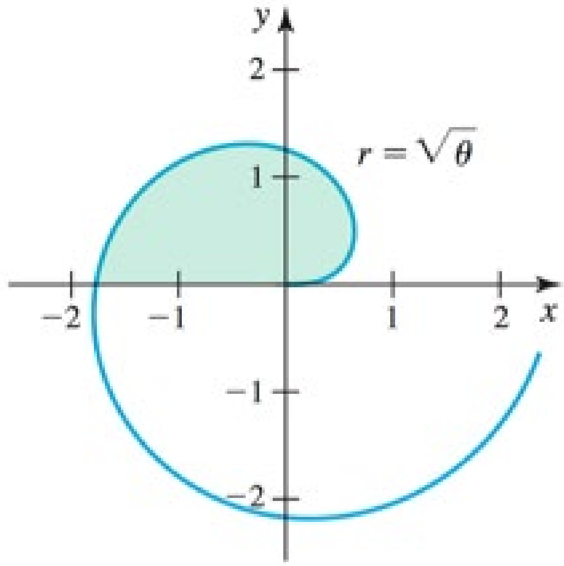
4. The Indian mathematician Srinivasa Ramanujan (1887-1920) discovered the following series representation for π:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Find the first four terms of the sequence of partial sums for the series and compute the resulting approximations to π. What is the speed of convergence of these approximations?

PLTL Calculus 2 Session 10 – Polar Area

The goal of this session is to get more familiar with polar integration and polar area, as well as to continue practicing with polar coordinates.



The basic formula for polar area is $A = \frac{1}{2}\int_a^b (f(\theta))^2\, d\theta$, where the curve is $r = f(\theta)$ and $a \le \theta \le b$. Use this setup to calculate the shaded area.

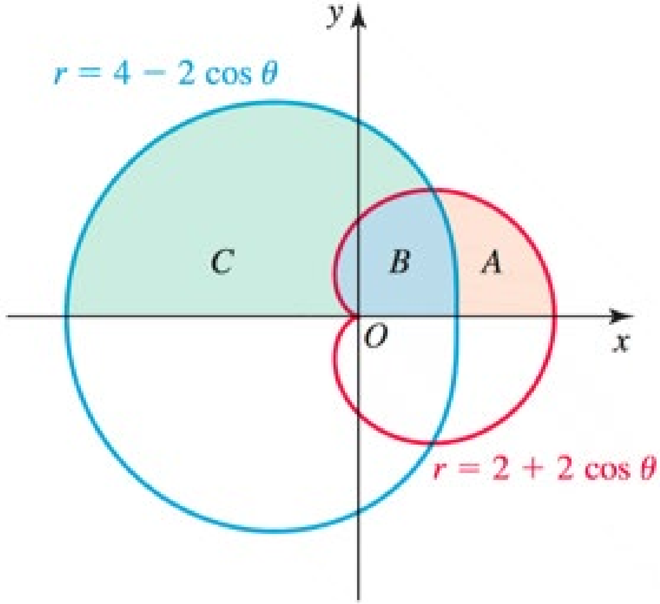
2. Next we need to be sure that we recognize how polar coordinates work so that we can solve polar equations. Explain why the point $(-1, 3\pi/2)$ is on the polar graph of $r = 1 + \cos\theta$ even though it does not satisfy the equation $r = 1 + \cos\theta$.

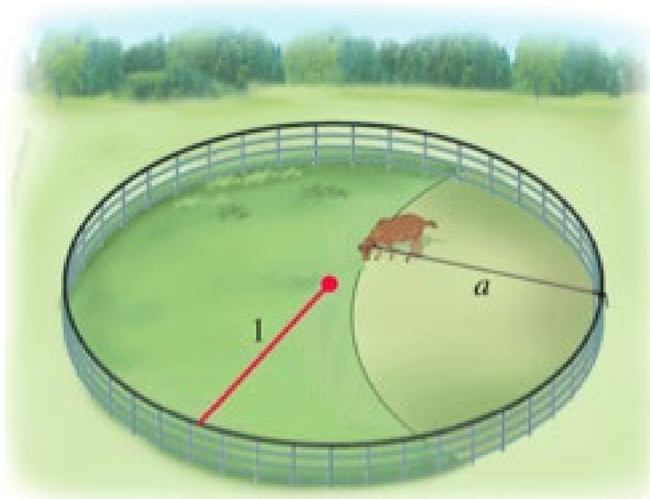
3. Now let's work on finding the area between two polar curves. The polar equation $r = 1/2$ is a circle of radius $1/2$ centered at the origin, and the equation $r = \cos\theta$ is a circle of radius 1 centered at the point $(1/2, 0)$.

(a) Sketch a graph of the two circles on the same set of axes and shade the region that is both outside the circle $r = 1/2$ as well as inside the circle $r = \cos\theta$.

(b) Determine the intersection points (r, θ) of the two circles.

(c) Integrate to find the area that you shaded in part (a).


y
$r = 4 - 2\cos\theta$
C
B
A
O
x
$r = 2 + 2\cos\theta$



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