

PLTL Calculus 1 Session 1 – Logarithmic Scales

Students are expected to read through the activity and do the problems that are highlighted **BEFORE** the live meeting with the peer leader on Sunday 9/12. The peer leader will discuss the other problems with you during the live session.

Logarithms are used to measure quantities that vary over many orders of magnitude. In this project, we explore two logarithmic scales used in science and engineering.

Background Knowledge

We will need to use that fact that logarithms and exponentials with the same base are inverses. In this project, we will only be dealing with the base $b = 10$. Here's what those inverse relationships look like:

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b(x)} = x$$

We will also need to use some rules of logarithms and exponentials, listed here:

- $b^{x+y} = b^x b^y$
- $b^{x-y} = b^x / b^y$
- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^r) = r \cdot \log_b(x)$

The Earthquake Magnitude Scale

The magnitude M of an earthquake is related to the released energy E by the following equivalent formulas:

$$\log_{10} E = 4.4 + 1.5M \quad \text{or} \quad E = (2.5 \times 10^4) \times 10^{1.5M}$$

where the energy E is measured in joules, and the magnitude M has no units. Using the second formula, the energy released by an earthquake can be thought of as a function of the magnitude; that is, we could write the equation as $E(M) = (2.5 \times 10^4) \times 10^{1.5M}$.

1. What is the magnitude of an earthquake that releases 4×10^{13} joules of energy?

2. Compute the ratios $E(2)/E(1)$, $E(3)/E(2)$, and $E(4)/E(3)$. Draw a conclusion about the value of $E(M + 1)/E(M)$ for any M . What does this mean about a one unit increase in magnitude, in terms of the energy released by an earthquake?
3. A one megaton nuclear bomb releases 5×10^{15} joules of energy. Compare that to the energy released by the following earthquakes:
- 1989 San Francisco, magnitude 7.1 (90 deaths)
 - 1975 Tangshan China, magnitude 7.9 (500,000 deaths)

The pH Scale

The acidity of a solution is measured by the concentration of positively charged hydrogen ions, which are hydrogen atoms that are missing their single electron. These hydrogen ions are denoted H^+ . The concentration of hydrogen ions is denoted $[H^+]$, and is usually measured in units of *moles per liter*; a **mole** is *Avogadro's number* of particles, which is about 6×10^{23} . The pH scale is defined by the following equivalent formulas:

$$pH = -\log_{10}[H^+] \quad \text{or} \quad [H^+] = 10^{-pH}$$

Pure water is **neutral** and has a pH of 7. **Acids** have a pH less than 7, and **bases** have a pH greater than 7.

4. What is the pH of a solution with a hydrogen ion concentration $[H^+]$ of 10^{-9} moles per liter? Is this solution an acid or a base?
5. In terms of hydrogen ion concentration $[H^+]$, how much more acidic is acid rain with a pH of 2 than ordinary rain with a pH of 6?

6. Assume that a lake polluted by acid rain has 100 million gallons of water with a pH of 4.
- What is the hydrogen ion concentration $[H^+]$ of the lake?
 - A chemical company dumps 100,000 gallons of acid with a pH of 2 into the lake. What is the new hydrogen ion concentration $[H^+]$ in the lake? What is the new pH after the acid is added?
 - Assuming the lake had the capacity, how much pure water (pH 7) would need to be added to the lake to restore its pH to 5.5?

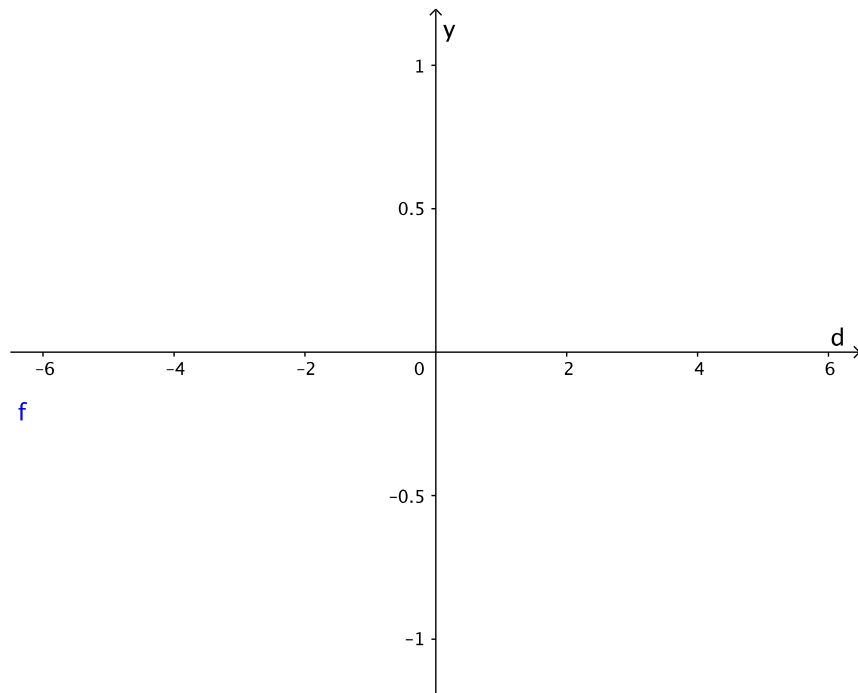
PLTL Calculus 1 Session 2 – Sine Waves

Students are expected to read through the activity and do the problems that are highlighted BEFORE the live meeting with the peer leader on Sunday Sept. 19. The peer leader will discuss the other problems with you during the live session.

Background Knowledge

Sine waves can be used to model periodic functions – that is, functions that repeat their pattern of values after a set interval length. In this project, we explore how to model the variation in sunlight at a specific latitude on earth throughout the year.

We will need to know the structure of a general sine wave. The formula is given below, and we're going to use the variable d in this project to be suggestive of the variable "day of the year." The basic sine wave graph $y(d) = \sin(d)$ looks like this:



Notice that the midline is at $y = 0$, the height above the midline is 1, and the graph repeats itself every 2π units. The zeroes are at all multiples of π .

As a generalization, for the function

$$y(d) = A \sin(B(d - C)) + D,$$

- the amplitude is $|A|$ - that's the height of the wave above the midline
- the period is $2\pi/|B|$ - that's how far you have to move for the graph to repeat itself
- there is a phase shift of C units to the right
- there is a vertical shift up D units

Let's figure out some specifics regarding this sine wave model $y(d) = A \sin(B(d - C)) + D$:

1. The midline of the wave $y(d)$ would be at $y =$ _____. The maximum value of $y(d)$ is $y =$ _____ and the minimum value is $y =$ _____.
2. What values of d make $y(d)$ equal to the midline value? (Hint: this is related to knowing the angles that make $\sin \theta = 0$.)
3. At what values of d does $y(d)$ achieve the maximum value? At what values of d does $y(d)$ achieve the minimum value? (Hint: this is related to knowing the angles that make $\sin \theta = 1$ and $\sin \theta = -1$.)

Constructing a Daylight Model

The amount of daylight on a given day is periodic with period equal to approximately 365 days. We will use the domain (input) variable d for day of the year; $d = 0$ corresponds to January 1, and $d = 365$ corresponds to December 31 (so we are assuming it is a leap year for this model). The output of our model should be hours of daylight on day d . The specific model for hours of daylight varies by location, based on the latitude of the location away from the equator. Indiana PA is roughly at 40° North latitude.

You know that the hours of daylight on a given day varies throughout the year; in Indiana, mid-summer days are very long, while mid-winter days are very short. The least amount of variation between the maximum and minimum amount of hours per day occurs at the equator, when every day has about 12 hours of daylight all through the year. On the other hand, the greatest amount of variation occurs at the North and South Poles, where the sun can go down and stay that way for months at a time in the Winter, but not set at all for months at a time in the Summer.

The longest day of the year in Indiana PA is June 21 (Summer Solstice), with about 15 hours of daylight. The shortest day of the year is Dec. 21 (Winter Solstice), with about 9 hours of daylight. The first day of Spring is March 21 (Vernal Equinox), and the first day of Fall is September 21 (Autumnal Equinox).

DO THIS:

Your task is to create a sine wave model $y(d) = A \sin(B(d - C)) + D$ that gives the correct amount of daylight on day d of the year in Indiana PA, where y is in hours and $1 \leq d \leq 365$. Once you have the model, use it to figure out the amount of daylight on September 16.

Make an attempt at this before the live session with the peer leader – you are not expected to get the answer beforehand, but you can try and think about it.

Sunrise/Sunset Models

You might expect that the shortest day of the year, December 21, has the latest sunrise time and the earliest sunset time... but this is not true! At 40° N latitude, the latest sunrise is at 7:25 a.m. on January 4, 14 days after the Winter Solstice, and the earliest sunset is at 4:37 p.m. on December 7, which is 14 days before the Winter Solstice.

Similarly, the earliest sunrise occurs at 4:30 a.m. on July 2, and the latest sunset is on June 7 at 7:32 p.m. These dates are 14 days after and 14 days before the Summer Solstice, respectively. So, even though the longest day of the year is on June 21, that is not the day with the earliest sunrise and latest sunset.

DO THIS:

Using sine functions, come up with a function $r(d)$ that gives the time of the sunrise on day d and a function $s(d)$ that gives the time of the sunset on day d . Assume that both functions are measured in minutes, and $r = 0$ and $s = 0$ correspond to 4:00 a.m. What can you say about the function $y(d)$ you created earlier in relation to these two functions $r(d)$ and $s(d)$?

CHALLENGE: A common misconception is that the amount of daylight we lose or gain each day is a constant. Actually, it varies throughout the year. Graph the daylight function $y(d)$ you created earlier, and use slope tools on your calculator to estimate the greatest amount of daylight lost or gained per day, and on what day(s) of the year that happens.

It will be helpful to set the viewing window on your calculator so that the input variable goes from 0 to 365, and the output variable has max and min matching your answers from #1 on page 2 of this activity.

PLTL Calculus 1 Session 3 - Limits

So far this month in class you covered ideas related to limits. Let's go over these topics one at a time, do some examples, and look at some applications. **Students should read the intro material for each section and attempt the highlighted problems before the live session with the peer leader.**

Infinite Limits occur when the y -values of a function $f(x)$ get arbitrarily large (either positive or negative) as x approaches a particular number a either from the right or from the left. This also means that $f(x)$ has a **vertical asymptote** at $x = a$. So, for a function $y = f(x)$ and a real number a , when we say we have an infinite limit we are dealing with at least one of the following cases:

$$\lim_{x \rightarrow a^-} f(x) = +\infty$$

$$\lim_{x \rightarrow a^+} f(x) = +\infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

Although it is technically true in each of these cases to say "the limit does not exist" and just write "DNE" instead of $\pm\infty$, it is always better to be more specific; if the limit does not exist, but you can specify that the y -values approach $+\infty$ or $-\infty$, you should do just that.

Example 1 The graph of f in the figure has vertical asymptotes at $x = 1$ and $x = 2$. Analyze the following limits:

1. $\lim_{x \rightarrow 1^-} f(x) =$

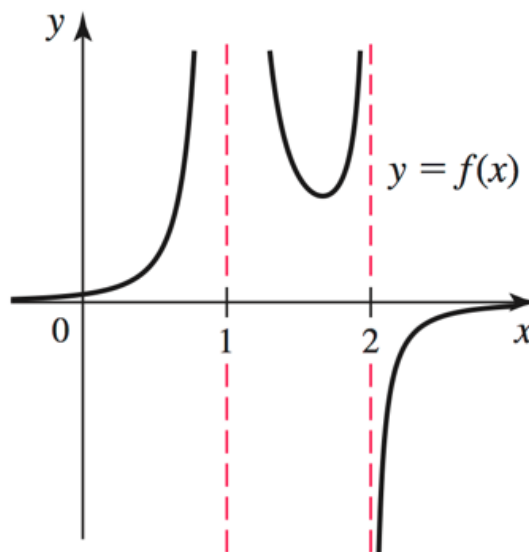
2. $\lim_{x \rightarrow 1^+} f(x) =$

3. $\lim_{x \rightarrow 1} f(x) =$

4. $\lim_{x \rightarrow 2^-} f(x) =$

5. $\lim_{x \rightarrow 2^+} f(x) =$

6. $\lim_{x \rightarrow 2} f(x) =$



Example 2 Verify that the function $g(x) = \frac{x^2 - 4x + 3}{x^2 - 3x + 2}$ is undefined at both $x = 1$ and $x = 2$. Does this mean that the graph of g has vertical asymptotes at both $x = 1$ and $x = 2$? Explain.

Example 3 Determining an infinite limit from a function formula can be harder than when you have a graph to look at. In both cases below, the function has a vertical asymptote at the x -value that is being approached. Determine if the limit is $-\infty$ or $+\infty$.

1. $\lim_{x \rightarrow 0} \frac{x - 3}{x^4 - 9x^2}$

2. $\lim_{\theta \rightarrow \pi/2^+} \tan(\theta)$

Example 4 Sketch a possible graph of a function f , together with vertical asymptotes, satisfying all the following conditions on $[0, 4]$: $f(1) = 0$; $f(3)$ is undefined; $\lim_{x \rightarrow 3} f(x) = 1$; $\lim_{x \rightarrow 0^+} f(x) = -\infty$; $\lim_{x \rightarrow 2} f(x) = +\infty$; $\lim_{x \rightarrow 4^-} f(x) = +\infty$.

Limits at Infinity means we are analyzing the behavior of the y -values of a function $f(x)$ as $x \rightarrow -\infty$ or as $x \rightarrow +\infty$. In these cases, we call the behavior of the y -values the **end behavior** of the graph of f . It's possible that these y -values become arbitrarily large as x becomes arbitrarily large (either positive or negative), and we end up with one of the following 4 cases:

$$\begin{array}{ll} \lim_{x \rightarrow -\infty} f(x) = +\infty & \lim_{x \rightarrow +\infty} f(x) = +\infty \\ \lim_{x \rightarrow -\infty} f(x) = -\infty & \lim_{x \rightarrow +\infty} f(x) = -\infty \end{array}$$

However, it's also possible that the y -values of the function f approach a real number as x gets arbitrarily large:

$$\lim_{x \rightarrow -\infty} f(x) = M \quad \text{or} \quad \lim_{x \rightarrow +\infty} f(x) = L$$

When either of these occur, we say that f has a **horizontal asymptote to the left** of $y = M$ and/or a **horizontal asymptote to the right** of $y = L$. (Make sure you can explain why a function can have 0, 1, or 2 horizontal asymptotes.)

Example 5 Determine the end behavior of the given function in both directions, and then list the horizontal asymptotes (if any).

1. $\frac{x^4 + 7}{x^5 + x^2 - x}$

3. $-3e^{-x}$

2. $\frac{3x^3 - 7}{x^2 - 5x^3}$

4. $\tan^{-1}(x)$

Application 1

If a function f represents a system that varies in time t , the existence of $\lim_{t \rightarrow \infty} f(t)$ as a real number means that the system reaches a **steady state**, or **equilibrium**. For the following systems, determine whether a steady state exists, and give the equilibrium value.

1. The population of rabbits on an island is given by $P(t) = \frac{3500t}{t+1}$
2. The amount of a drug (in milligrams) in the blood after an IV tube is inserted is given by $D(t) = 200(1 - 2^{-t})$
3. The amplitude of a spring oscillator is given by $A(t) = 2 \left(\frac{t + \sin(t)}{t} \right)$

Continuity is a concept that means to be unbroken. From a limit standpoint, that means that for a function $f(x)$ to be continuous at a point $x = a$ there must exist a real number L so that:

$$\begin{aligned} (1) \quad & f(a) = L \\ (2) \quad & \lim_{x \rightarrow a^-} f(x) = L \\ (3) \quad & \lim_{x \rightarrow a^+} f(x) = L \end{aligned}$$

In other words, the function must exist, the limit must also exist, and the function value and limit must be equal to each other. Here's why continuity is your best friend when working with limits: if a function is continuous at a point $x = a$, then you can evaluate $\lim_{x \rightarrow a} f(x)$ simply by evaluating $f(a)$.

Example 6 Use continuity to evaluate each of the following limits:

1. $\lim_{x \rightarrow 1} \left(\frac{x+7}{x-5} \right)^4$

2. $\lim_{\theta \rightarrow \pi} \cos^2(\theta) - 6 \sin(\theta)$

Example 7 A good way to make sure you know the definition of continuity is to work out the answer to this problem. Determine the value(s) of the constant k that make $f(x)$ continuous at $x = 0$ if

$$f(x) = \begin{cases} \frac{x+k}{x+3} & x < 0 \\ \frac{100-x}{50} & x \geq 0 \end{cases}$$

There are three main ways that a function f could be discontinuous at a point $x = a$:

1. f has a vertical asymptote at $x = a$. This is usually signified by the function being of the form $\#/0$ when you plug in $x = a$.
2. f has a hole at $x = a$. This is usually a possibility when the function has the form $0/0$ when you plug in $x = a$.
3. f has a jump at $x = a$. This is a possibility when f is defined with a piecewise formula.

Example 8 Determine the interval(s) on which the following functions are continuous:

1. $\frac{x^2 - 4x + 3}{x^2 - 1}$

2. $\csc(x)$

Application 2

A basic problem that comes up all the time in science is finding solutions of the equation $f(x) = R$. Sometimes these equations are complicated and finding solutions is computationally expensive, so it is worthwhile to apply a simple consequence of continuity called the **Intermediate Value Theorem** to determine if solutions exist at all. Here's what the Theorem says:

Suppose f is continuous on the interval $[a, b]$ and R is any number between $f(a)$ and $f(b)$. Then there exists at least one number $x = c$ in the interval (a, b) satisfying $f(x) = R$.

Example Assume you invest \$250 at the end of each year for 10 years at an annual interest rate r . The amount of money in your account after 10 years is given by

$$A(r) = \frac{250((1+r)^{10} - 1)}{r}$$

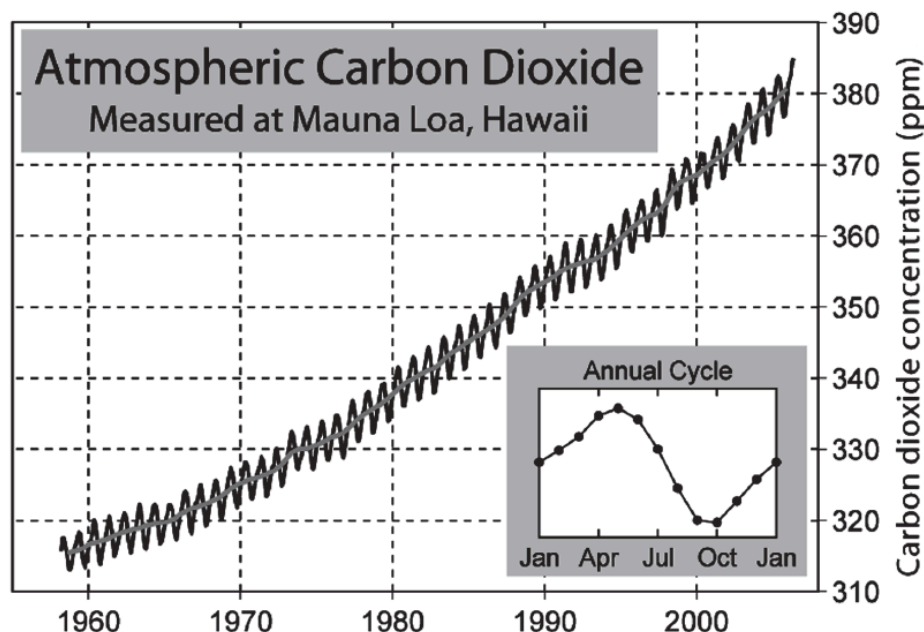
Assume your goal is to have \$3500 in your account after 10 years. Show there is an interest rate r between 1% and 10% that allows you to reach your financial goal, and estimate the value of r to four decimal places.

PLTL Calculus 1 Session 4 – CO₂ Emissions

Students are expected to read through the activity and do the problems that are highlighted BEFORE the live meeting with the peer leader. The peer leader will discuss the other problems with you during the live session.

(Note: This project is an existing accompaniment activity to your e-book, *Calculus Early Transcendentals* by Briggs et al, published by Pearson.)

The data shown in the figure below have become a cornerstone in the debate about global warming. Recorded at the Mauna Loa Observatory in Hawaii, the data give monthly measurements of atmospheric carbon dioxide (CO₂) in parts per million over a 50-year period.



In this project we investigate various approaches to describing the data with functions. The data have two significant features: the graph rises steadily over time, but there is also a regular sequence of oscillations as shown in the magnification bump out graph. We want to come up with a function formula, or model, that closely fits this data.

1. Why would scientists want to determine a formula that models the data?

Oscillations

2. What is the period and amplitude of the oscillations? (The period would be the length of time between the peaks, and the amplitude is half the distance from a peak to the following valley.)

3. Assume that t is measured in years and assume that $t = 0$ corresponds to the year 1960. Use your answers to #2 and what you learned in last week's activity to write a function in the form $O(t) = A \sin(Bt)$ that describes only the oscillations in the data. Then graph the function from 1960 to 2010, giving the window dimensions of your graph to best fit the graph on the screen.

Rising Base Curve

4. The steadily increasing base curve in the picture given above corresponds to the average of the monthly data over a one-year period. What sort of function would give a reasonable fit to this base curve? Why do you think that?

5. Two points from the data set are $(0, 317)$ corresponding to the year 1960 and $(40, 370)$ corresponding to the year 2000. Find an equation of the line that passes through the points and plot that line on the given graph. How well does the line fit the base curve?

6. Find a different pair of points than the pair given in #5 that better fits the data. Find an equation of the line that passes through those points instead. Call this linear function $L(t)$.

7. Another possible form for the increasing base function would be an exponential function of the form $E(t) = Ce^{kt}$, where C and k are constants. Use the data points $(10, 325)$ and $(30, 354)$ to determine an exponential function that fits the base curve.

Combined Curve

8. We can model a curve that closely fits the data as in the given picture by combining the oscillation function $O(t)$ from #3 with the linear function $L(t)$ from #6. Define $C_1(t) = O(t) + L(t)$ as a CO₂ function that fits the full data set. Graph $C_1(t)$ over the interval $0 \leq t \leq 60$, which includes a projection of the data through the year 2020, and write down the predicted amount $C_1(60)$ of CO₂ for 2020.

9. We can also model a curve that closely fits the data as in the given picture by combining the oscillation function $O(t)$ from #3 with the exponential function $E(t)$ from #7. Define $C_2(t) = O(t) + E(t)$ as a CO₂ function that also fits the full data set. Graph $C_2(t)$ over the interval $0 \leq t \leq 60$, which includes a projection of the data through the year 2020, and write down the predicted amount $C_2(60)$ of CO₂ for 2020.

10. Check out <https://www.esrl.noaa.gov/gmd/ccgg/trends/> to see the actual amount of CO₂ recorded at the Mauna Loa Observatory in January 2020. Which model was more accurate? What factors might explain the discrepancy?

PLTL Calculus 1 Session 5 – Local Linearity

Today we are going to use limits to help us approximate the slopes of curves at certain points. You have spent multiple weeks learning about how to calculate limits, and this activity is a major application of limits that you have started exploring in class as well.

Consider the graph of $f(x)$ and its **tangent line** at the point $(a, f(a))$ as in Figure 1a below. The tangent line to a curve $f(x)$ at a point is the line that just touches the curve at that point and has the same direction as the curve at that point. Because the graph is smooth at $(a, f(a))$ (meaning that it has no sharp corners), $f(x)$ looks and acts like the tangent line to $f(x)$ at $x = a$, as long as we stay close to that point. In fact, if we zoom in further on the curve near $(a, f(a))$, the line and the curve become nearly indistinguishable as in Figure 1b below. For this reason, when we refer to the slope of the curve $f(x)$ at the point $(a, f(a))$, we mean the value of the slope of the tangent line to $f(x)$ at that point – they are interchangeable.

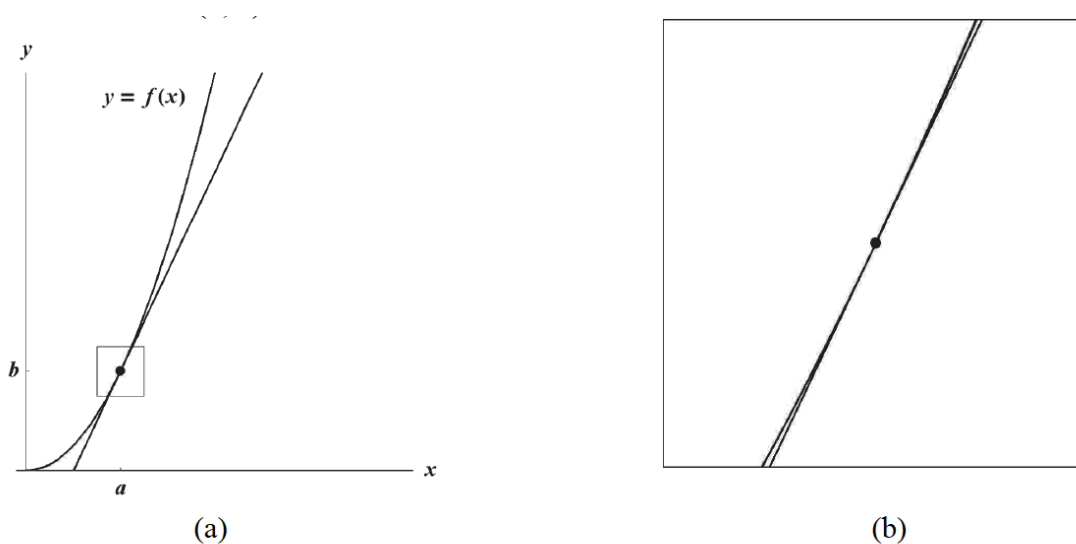


Figure 1

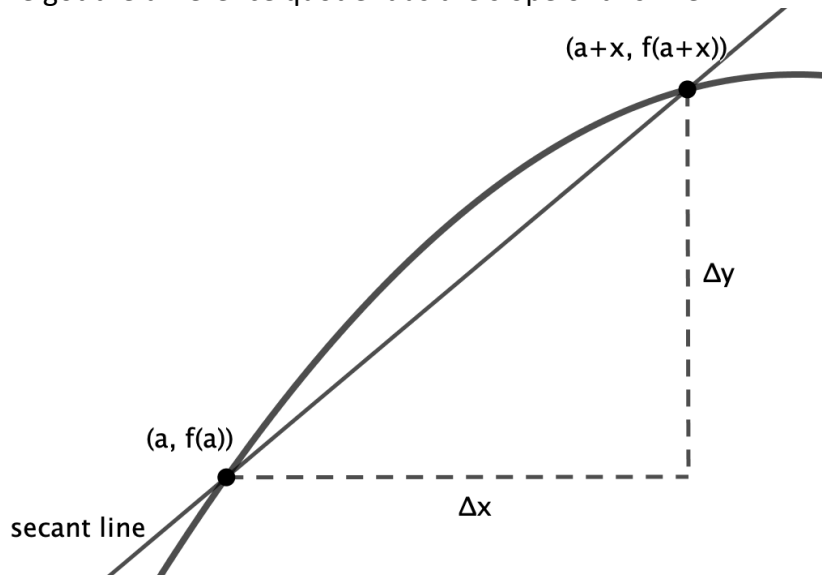
In this project, we will consider two different ways to estimate the slope of a smooth curve $f(x)$ at the point $(a, f(a))$:

1. Use a graphing utility to zoom in on the graph of $f(x)$ closer and closer to the point $(a, f(a))$ until the curve appears linear, and use points on the curve to approximate the slope of the tangent line to $f(x)$ at that point
2. Use the limit as $x \rightarrow a$ of the **difference quotient**

$$\frac{f(a + x) - f(a)}{x},$$

which gives the slope of the secant line to $f(x)$ containing the points $(a, f(a))$ and $(a + x, f(a + x))$. (In class, you will use the variable h instead of x to represent the small distance between the two x -values of the points – we are using x here because we are going to enter the difference quotient into our calculator, and in many cases calculators only accept the variable x as input).

Here's a look at the secant line we are talking about – make sure you understand how we got the difference quotient as the slope of this line.



Background Knowledge

Things we need to know to work on this project:

- On the graphing calculator or in another graphing utility, how to enter a function, graph that function on a particular viewing window, use the Trace feature in the graphing window, view a Table of values for a function, and change settings for Tables.
- The slope of a line containing the points $(a, f(a))$ and $(b, f(b))$ is

$$m = \frac{f(b) - f(a)}{b - a}$$

- The equation of a line with slope m passing through the point $(a, f(a))$ is

$$y - f(a) = m(x - a)$$

- The meaning of the expression

$$\lim_{x \rightarrow a} E(x) = L$$

is that the expression $E(x)$ has y -values that get closer and closer to $y = L$ as the x -values get closer and closer to $x = a$.

1. Consider the function $y = \sqrt{x}$ at the point $(4,2)$.

- Enter the function $y = \sqrt{x}$ as Y2 in your calculator and plot the graph on the viewing window $[0,8] \times [0,4]$ (this notation means that $0 \leq x \leq 8$ and $0 \leq y \leq 4$). Notice that the graph is definitely not linear at this viewing level, so we need to zoom in.
- Plot the graph instead on the viewing window $[3,5] \times [1.5,2.5]$. The curve now should appear to be close to linear. Use the Trace command to find the points on the graph for $x = 3.5$ and $x = 4.5$ (when using Trace, you can manually enter the exact x -values you want). What is the slope of the line that contains those two points of the graph of the curve? Use 8 digits of accuracy for the y -coordinates and slope, and put your answers in the first row of the table below.

Graphing Window	Points on Curve	Slope
$[3,5] \times [1.5,2.5]$	(3.5,) (4.5,)	
$[3.5,4.5] \times [1.7,2.2]$	(3.75,) (4.25,)	
$[3.75,4.25] \times [1.9,2.1]$	(3.9,) (4.1,)	
$[3.9,4.1] \times [1.95,2.05]$	(3.95,) (4.05,)	

- Continue to fill out the table as you zoom in more and more on the point $(4,2)$. For each interval, we have picked a pair of points, with one point to the left of $(4,2)$ and the other point to the right of $(4,2)$.
- What would be a reasonable estimate of the slope of $y = \sqrt{x}$ at the point $(4,2)$ based on the table values? Use this slope estimate to write an equation for the tangent line to $y = \sqrt{x}$ at the point $(4,2)$, and then enter the equation for the tangent line as function Y3 in your calculator. Graph the function and tangent line together on the original viewing window $[0,8] \times [0,4]$.

- e. Now we will examine the same question from the standpoint of the limit on the values of the difference quotient. Write down the slope of the secant line to $y = \sqrt{x}$ containing the points $(4,2)$ and (x, \sqrt{x}) . Enter this as function Y1 in your calculator.

slope of secant =

- f. In the Table Settings, make sure that the independent variable is set to 'Ask' instead of 'Auto'.
- g. In the Table window, enter the sequence of x -values you listed for the points to the LEFT of $(4,2)$ in the table from part (c). What value does Y1 seem to be approaching as $x \rightarrow 4$?
- h. In the Table window, enter the sequence of x -values you listed for the points to the RIGHT of $(4,2)$ in the table from part (c). What value does Y1 seem to be approaching as $x \rightarrow 4$?
- i. Compare your answers in parts (g) and (h). What would be a good estimate for the slope of the tangent line to $y = \sqrt{x}$ at the point $(4,2)$? Is this the same as your answer to part (d)? Why or why not?

2. Next consider the function $y = |x|$.
- a. Graph $y = |x|$ on the viewing window $[-5,5] \times [-1,5]$. Explain why the function is not locally linear at the origin. What does this mean about the tangent line to $y = |x|$ at $x = 0$?
- b. The graph of $y = |x|$ IS locally linear away from the origin. Write equations for the tangent lines to $y = |x|$ for $x < 0$, and equations for the tangent lines to $y = |x|$ for $x > 0$. (Hint – using the piecewise definition of the absolute value function will save you from doing any work like you had to do in Problem #1 above.)

3. Now consider the piecewise defined function

$$f(x) = \begin{cases} ax + b, & \text{if } x \leq 4 \\ \sqrt{x}, & \text{if } x > 4. \end{cases}$$

- a. Use a graphing utility to graph f on the viewing window $[0,8] \times [0,4]$ with $a = 2$ and $b = -6$. Is the function locally linear at $x = 4$ with this choice of a and b ? (locally linear means smooth – in order to achieve this, the slope from the left and the slope from the right should match up.)

- b. Repeat part (a), but this time use $a = 1/2$ and $b = 0$. Now is the function locally linear at $x = 4$?

- c. Determine the values of a and b that make f locally linear at $x = 4$. (Hint - you will want to revisit the work you did for Problem #1 above.)

PLTL Calculus 1 Session 6 – Drug Rates

Students are expected to complete the two parts of the activity that are highlighted before going to the live peer-led session on Sunday.

When a drug is administered intravenously it enters the bloodstream immediately, producing an immediate effect for the patient. The drug can either be given as a single rapid injection or at a constant drip rate (IV infusion). Common drugs administered intravenously include morphine for pain, diazepam to control a seizure, and digoxin for heart failure. In this activity, we will explore the differences between the two types of drug administration, and then calculate the rate of change of the drug in the bloodstream in each case.

Background Knowledge

Things we need to know to work on this project:

- How to solve an exponential equation using logarithms
- Exponential growth and exponential decay can always be modeled by a function of the form $A(t) = De^{kt}$ (or a transformation of this model), where D is the initial dose of the drug and k is the growth or decay constant for the drug model.
If k is positive, the model is for exponential growth; if k is negative, the model is for exponential decay.
- The slope of a line containing the points $(a, f(a))$ and $(b, f(b))$ is

$$m = \frac{f(b) - f(a)}{b - a}$$

Remember that this can also be viewed as the slope of the secant line to the function f between the two points, and also can be interpreted as the average rate of change of f over that interval.

First type: Single Rapid Injection

With a single rapid injection, the amount of the drug in the bloodstream reaches its peak immediately, and then the body eliminates the drug exponentially. The drug labetalol is used for the control of blood pressure in patients with severe hypertension. The half-life of labetalol is 4 hours (this means that if you measure the amount of drug in the bloodstream at any particular time, there will be half that amount in the bloodstream 4 hours later). Suppose that an initial dose of 35 mg is administered to a patient by rapid injection.

1. Find a model of the form $A(t) = De^{kt}$ for the amount of drug in the bloodstream t hours after the drug is administered. You will need to use the information about the half-life to figure out the value of the constant k . Round the value of k to 3 decimal places. Graph the function on the viewing window $[0,24] \times [0,40]$ and observe the end behavior.
2. Find the average rate of change of the drug in the bloodstream, with units, for the time interval from $t = 0$ to $t = 2$, and then again for the time interval from $t = 4$ to $t = 6$. Make an observation about these rates of change in relation to the half-life of the drug; then, without calculating, estimate the average rate of change of the drug in the bloodstream over the time interval from $t = 8$ to $t = 10$.

3. In this step, we will calculate the instantaneous rate of change at $t = 4$; this would be the exact rate at which the body is expelling the drug from the bloodstream after 4 hours. Write down the difference quotient for A giving the rate of change from the point $(4, A(4))$ to the general point $(x, A(x))$. Then, using a calculator, make a table of values for the difference quotient for a successive list of x values moving closer and closer to 4. Finally, make a reasonable assumption about the limit of these table values.
4. Using your observations in parts 2 and 3 above, write down the instantaneous rate of change of the amount of drug in the bloodstream at $t = 8$ without doing any calculations. Be sure you can explain why you were able to do this.

5. An advantage of rapid injection is that the amount of drug in the bloodstream is immediately an effective amount; however, a drawback is that repeat injections have to be made at carefully calibrated time intervals and amounts to keep an effective amount of drug in the bloodstream. Assuming that the drug is only effective for the patient as long as there are 20 mg or more in the bloodstream, after how many hours would the patient need a second dose? If the maximum safe amount of the drug in the body is 35 mg, what should the second dose be?

Second type: Intravenous (IV) Infusion

With an IV infusion, the amount of drug in the bloodstream starts at zero and increases until the rate the drug is entering the body equals the rate the drug is being eliminated from the body; at this point, the amount of drug in the bloodstream levels off. This type of situation can be modeled by a limited growth exponential function of the form

$$A(t) = -\frac{r}{k}(1 - e^{kt})$$

where r is the rate of infusion per hour and k is the exponential decay constant of the drug (which is the same as it was for the single rapid injection).

Say that we administer the same drug (which still has a half-life of 4 hours) at a constant drip rate of 6 mg/hr.

1. Find a model for the amount of drug in the bloodstream t hours after the IV infusion begins. Graph the function on the viewing window $[0,24] \times [0,40]$ and observe the end behavior.

- Find the average rate of change of the drug in the bloodstream, with units, for the time interval from $t = 0$ to $t = 2$, and then again for the time interval from $t = 4$ to $t = 6$. Make an observation about these rates of change in relation the rates of change you calculated for the single rapid injection; use this observation to write down the instantaneous rate of change of the IV infusion drug at time $t = 4$.
- An advantage of an IV infusion is that a dose can be given such that the limit of $A(t)$ as t approaches infinity is an effective level, and once the amount of the drug reaches this level it will remain there as long as the infusion continues. However, it takes a comparatively long amount of time for the amount of the drug to reach the desired level. For this IV infusion, how long would it take before the amount of drug in the body was greater than or equal to the minimum effective amount of 20 mg?

Challenge

Can you find a drug administration method that will immediately administer 35 mg of the drug, and keep that amount constant?

PLTL Fall 2021 Calculus 1 Session 7 – Elasticity and Economics

Economists use the term elasticity in relation to supply, demand, income, labor, capital, and other variables in systems with input and output. Basically, elasticity describes how changes in the input to a system are related to changes in the output.

If that sounds familiar, it is because differentiation accomplishes the same task: measuring how a small change in the input variable affects change in the output variable.

Background Knowledge

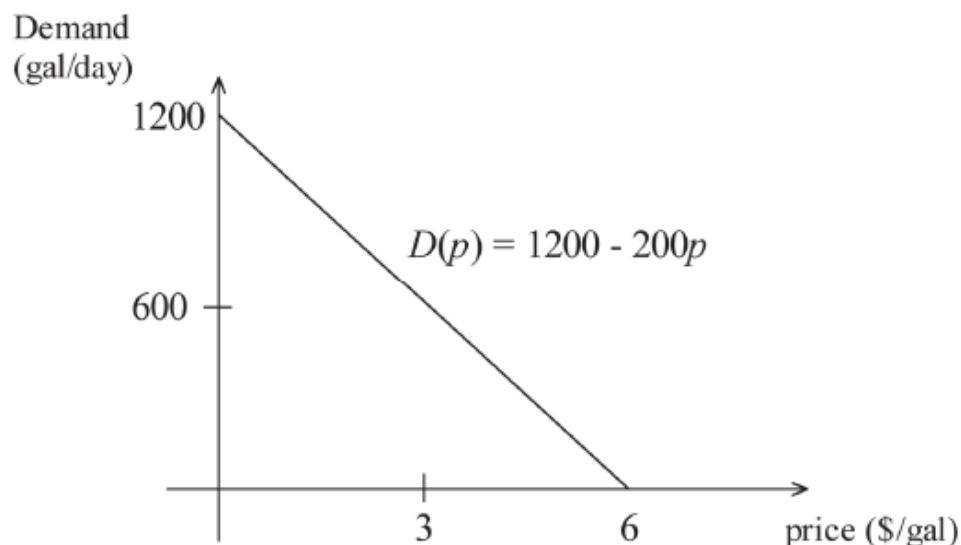
In order to be able to work through this activity, you should be familiar with the differentiation shortcut rules you learned in class: constant, constant multiple, sum/difference, power, general exponentials, product, quotient.

You should also be familiar with the idea that a derivative measures rate of change.

Linear demand

Normally, as the price of an item increases, the number of sales of that item generally decreases. This relationship is expressed in a demand function $D(p)$ that describes how demand depends on price.

For this example, suppose a gas station has the demand function $D(p) = 1200 - 200p$ for price p ; the graph is shown here.



1. According to this model $D(p) = 1200 - 200p$, how many gallons of gas can the gas station owners expect to sell per month if the price is set at \$4 per gallon? What about if the price is \$3.50 per gallon? Which of these price points results in greater revenue for the gas station?
2. Evaluate $D'(p)$ and explain why your answer means that the demand function is decreasing. Why would you expect that demand functions should usually be decreasing functions?
3. For the comparison we made in Step 1, call the change in price $\Delta p = 4.00 - 3.50$; call the resulting change in the number of gallons sold ΔD , and write down this amount. Now, express your answers in terms of *percentages*: What is the *percent change* in price $\Delta p/p$, and what is the resulting *percent change* in the number of gallons sold $\Delta D/D$?

4. The elasticity in the demand, E , is the ratio of the percent change in demand to the percent change in price:

$$E = \frac{\Delta D / D}{\Delta p / p}$$

Compute the elasticity for the amounts of change from Step 3.

The elasticity is simplified by considering only very small changes in p and D . In this case, we can apply the definition of the derivative:

$$E = \lim_{\Delta p \rightarrow 0} \frac{\Delta D / D}{\Delta p / p} = \lim_{\Delta p \rightarrow 0} \frac{\Delta D}{\Delta p} \left(\frac{p}{D} \right) = \frac{dD}{dp} \left(\frac{p}{D} \right)$$

Prove that elasticity is a function of p only, and for this gas station example find a formula for $E(p)$. Evaluate $E(3.5)$ and compare to your answer in Step 4.

5. The elasticity may be interpreted as: *the percent change in the demand that results from every one percent change in the price*. For example, if $E(p) = -2$, a one percent increase in price results in a two percent decrease in demand.

For our gas example here, if the price of gasoline is \$4.50 and there is a 3.5% increase in the price, what is the elasticity and the corresponding percent change in the number of gallons of gas sold? Would this move be profitable for the gas station?

6. When $-\infty < E < -1$, the demand is said to be elastic, and when $-1 < E < 0$, the demand is said to be inelastic. Further, when $E = -\infty$ the demand is perfectly elastic, and when $E = 0$ the demand is perfectly inelastic. Essential goods such as basic food tend to have inelastic demand, but discretionary items such as electronics tend to have elastic demands. Using these examples, explain why you think we use the terms *elastic* and *inelastic*.

7. Graph the gasoline demand elasticity function for $0 \leq p < 6$. For what prices is the gasoline demand function elastic? For what prices is the gasoline demand function inelastic?
8. As another example, the demand for processed pork in Canada is described by the function $D(p) = 286 - 20p$. Graph this demand function, compute the elasticity, and graph the elasticity. For what prices is the demand function elastic? For what prices is the demand function inelastic?

General Examples

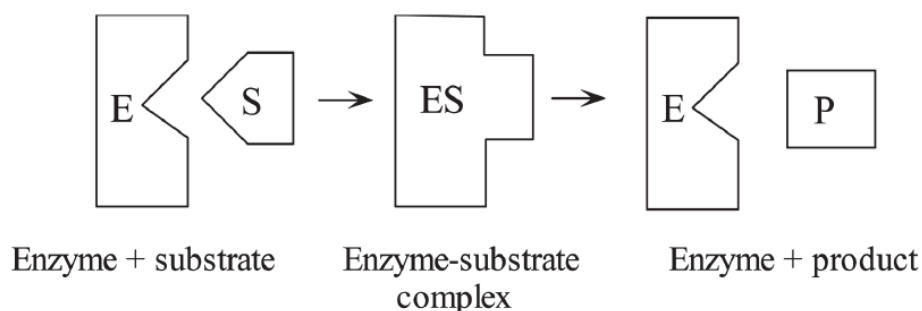
1. Show that if a and b are positive real numbers, the general linear demand function $D(p) = a - bp$ has decreasing elasticity for $0 \leq p < a/b$. What is true about the elasticity when $p = a/b$?
2. Compute the elasticity for the exponential demand function $D(p) = ae^{-bp}$, where a and b are positive real numbers. For what values of p is the demand elastic? Inelastic? What does that mean?
3. Compute the elasticity for the demand function $D(p) = a/p^b$, where a and b are positive real numbers. For what values of p is the demand elastic? Inelastic? What does that mean?

PLTL Calculus 1 Session 8 – Enzyme Kinetics

Students should read over the introduction, and then attempt the highlighted parts of the activity before the Sunday evening session.

Enzymes are catalysts that facilitate the biochemical reactions that occur within all living organisms. One of the fundamental laws of enzyme kinetics was proposed by Loenor Michaelis and Maud Menten in 1913. The law has been supported through laboratory experiments and explained through mathematical modeling; today, Michaelis-Menten kinetics are used in many biological models.

An enzyme molecule is designed to “fit” another molecule called a *substrate*. The substrate (S) and enzyme (E) form an intermediate complex (ES), which then dissociates to form the final end-product of the reaction (P) and the original enzyme. The enzyme can then be re-used. This process is shown in the following diagram.



Background Knowledge

- The rate of change of a function at a point is the derivative of the function there.
- Functions are increasing when they have positive slope and decreasing when they have negative slope. You can identify where a function is increasing and decreasing by finding where the derivative is zero, and then making a sign diagram for the intervals determined by those zeroes.
- Graphing functions requires manipulation of the graphing window so that you can see all the graph features, including end behavior.

An important question concerns the rate at which product molecules are formed. Under certain assumptions, Michaelis-Menton kinetics relates the rate of production of P to the amount of substrate present. In this project, we will explore these production rates, along with looking at how small changes in each variable changes the overall reaction over time. Let R be the rate of production of the final product P , and let s be the concentration of the substrate initially present. Both s and P are measured in units such as micro-moles μM , while R is measured in $\mu\text{M/s}$.

The Michaelis-Menton law says that

$$R(s) = \frac{Vs}{K + s},$$

where $V > 0$ and $K > 0$ are constants that are specific to each enzyme.

1. Let $K = 5 \mu\text{M}$ and $V = 10 \mu\text{M/s}$. Graph R as a function of s on the window $[0, 35] \times [0, 12]$. The graph looks to be increasing; prove that it actually is always increasing, using R' , for this particular choice of V and K .
2. What does the function R mean biologically? As the initial concentration of the substrate s increases, what is the effect on the production P ? (Remember, P is the actual amount of product produced, and R is the rate of change of P .)

Now let's interpret the constants V and K .

3. Evaluate the limit as $s \rightarrow \infty$ of $R(s)$. Explain why V is the maximum production rate. Is there any value of s for which the production rate equals V ?
4. How does the shape of the graph of R change if V increases? How does it change if V decreases?
5. The constant K has the same units as s . Evaluate $R(K)$, the rate of production when $s = K$. Show that K is the initial enzyme concentration that gives a production rate of $V/2$ (so, half the maximum rate).

6. How does the shape of the graph of R change if K increases? How does it change if K decreases? Explain why a small value of K means that the enzyme has a high affinity for the substrate.

7. Evaluate $R'(0)$, which is the slope of the curve at the origin. How does the slope change if K is increased with V fixed? How does the slope change if V is decreased with K fixed?

8. Now suppose a particular enzyme obeys Michaelis-Menton kinetics, but we do not know the exact values of the parameters V and K . Now suppose that we measure two data points: $(s_1, R_1) = (2 \mu\text{M}, 0.5 \mu\text{M/s})$ and $(s_2, R_2) = (5 \mu\text{M}, 1 \mu\text{M/s})$. Find values of K and V that fit this data, and plot the points and the production rate function R in this case.
9. Occasionally, the Michaelis-Menton law is graphed with a semilog plot, which means that $\ln(s)$ is the variable for the horizontal axis. Mke a semilog plot of the Michaelis-Menton Law with $K = 5 \mu\text{M}$ and $V = 10 \mu\text{M/s}$.

PLTL Calculus 1 Session 9 – Newton's Method for Approximating Roots

Students should read through the introduction and attempt the highlighted parts of the activity before the live Sunday session with the peer leader.

One of the most important parts of the process of finding maximums and minimums is solving equations of the form $g'(x) = 0$ to find some or all of the critical points of the function $g(x)$.

In this project, we are going to explore a specific numerical method for approximating the roots of a function $f(x)$, which is the same as approximating the solutions of an equation of the form $f(x) = 0$.

Background Knowledge

- The derivative $f'(x)$ tells us the slope of the tangent line to the graph of $f(x)$ at any point where $f(x)$ is differentiable; so, $f'(x_0)$ is the slope of the tangent line to $f(x)$ at the point where the x -coordinate is x_0 .
- The equation of the tangent line to $f(x)$ at the point $(x_0, f(x_0))$ is
$$y - f(x_0) = f'(x_0)(x - x_0)$$
which can be solved for y and then has the form
$$y = f(x_0) + f'(x_0)(x - x_0).$$

Newton's Method

Assume that r is a solution of $f(x) = 0$, as shown in Figure 1 below, so that $f(r) = 0$.

Our goal is to approximate the value of r to any degree of accuracy by obtaining closer and closer approximations, starting with the initial approximation x_0 that is also shown in Figure 1.

To obtain a (hopefully) better approximation to r , two steps are carried out:

- The point x_1 where the tangent line intersects the x -axis is found; this new point is our replacement approximation to the value of r , as shown in Figure 2 as well.

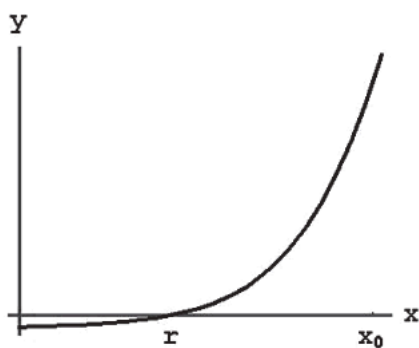


Figure 1

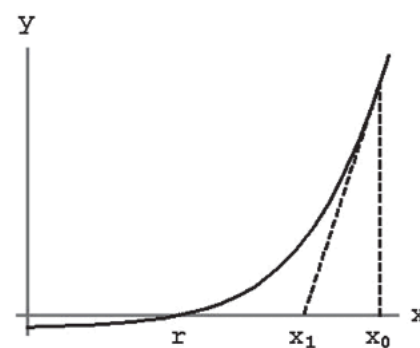


Figure 2

Notice that x_1 is a better approximation to r than x_0 (in this case, anyway).

Next, to improve upon the new approximation x_1 , we repeat the two-step process outlined above to get the next approximation x_2 using the tangent line to $f(x)$ at x_1 , as shown in Figure 3 below.

Continuing like this, we get a sequence of approximations $\{x_0, x_1, x_2, x_3, \dots\}$ that ideally get closer and closer to the root r ; in general, after n steps, the line tangent to $f(x)$ at x_n is used to get the next approximation x_{n+1} in the sequence (see Figure 4).

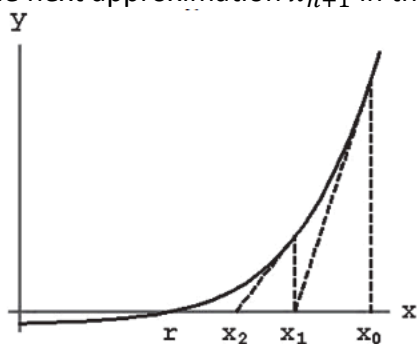


Figure 3

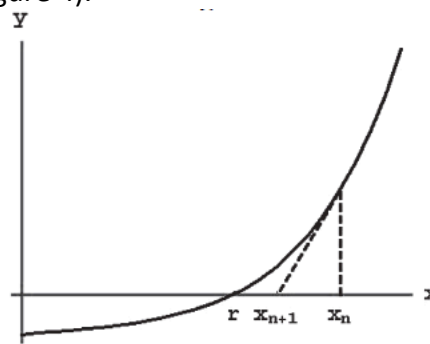


Figure 4

Let's derive a specific formula for the new approximation x_{n+1} based on the previous approximation x_n :

1. Find the equation of the tangent line to a general function $f(x)$ at the point x_n , and solve the equation for y .
2. Assuming that the tangent line you just found intersects the x -axis at the point x_{n+1} , as in Figure 4, verify that

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}.$$

3. Now we will apply Newton's Method to find a root of $f(x) = x^3 - 5x + 1$.

a. Show that Newton's method takes the form

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 1}{3x_n^2 - 5}$$

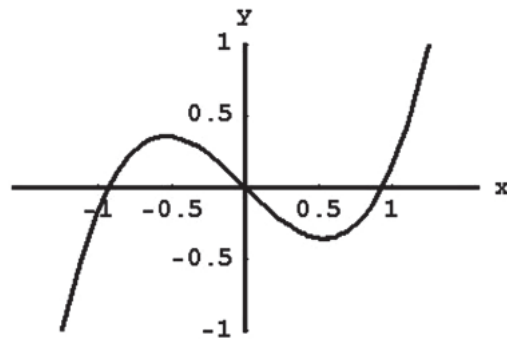
b. Assuming $x_0 = 3$, find the values of successive approximations x_1, x_2, x_3 using the formula from part (a). What is your approximation of a root of $f(x)$?

c. Assuming $x_0 = 1$, find the values of successive approximations x_1, x_2, x_3 using the formula from part (a). What is your approximation of a root of $f(x)$ this time? Look at the graph of $f(x)$, and explain why the root is different in case (b) as compared to case (a).

4. Now find an approximate solution of $x^3 - \cos(x) = 0$ using initial approximation $x_0 = 2$, iterating Newton's Method enough times so that successive approximations agree to six decimal places.

5. Let $f(x) = 0.2x^5 - x + 1$.
- Use Newton's Method with $x_0 = -1.5$, approximate a solution of $f(x) = 0$ with six decimal places of accuracy.
 - Why would Newton's Method fail to home in on a root if our initial approximation is $x_0 = -1$? It may help to graph $f(x)$ on the window $[-3, 2] \times [-2, 5]$.

6. It is extremely important to make a good choice of the initial approximation x_0 to the root r ; a poorly chosen value of x_0 can lead to unexpected results. Consider this graph of $f(x) = x^3 - \sin(x)$, indicating that there are three roots of $f(x)$:



You can check that $x = 0$ is one root, and it looks like the other two roots are near $x = 1$ and $x = -1$.

- a. Suppose we wish to verify that Newton's Method approximates the known root $x = 0$ by using an initial value of $x_0 = 0.49$. Calculate the iterated approximations x_1, x_2, x_3, \dots until two consecutive values agree to six decimal places. What happens and why?

- b. What happens if we change the starting value to 0.4 instead? What about 0.6?

PLTL Calculus 1 Session 10 - Optimization Problems

The goal of this session is to take applied problems seeking extreme values, and then use what we have learned about finding maximums and minimums to solve the problem.

In each case, we have an **objective function**, which is the quantity we want to optimize, and it is subject to one or more **constraints**, which are conditions that limit the possible values of the variables involved.

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Be very careful about step 6 in the guidelines! When you find a critical point for the objective function, this is not automatically the answer, and you are not done with the problem – you have to check that the critical point actually gives you the maximum or minimum value that you are looking for. You wouldn't want to report a value that gives a MINIMUM when you are actually looking for a MAXIMUM!

Also, you want to make sure that once you find a critical point that optimizes the objective function, you go back and read the question to determine what form your answer should have. For example, if your critical point measures the length of a container, is the question asking about the volume of that container instead?

In order to check whether a critical point gives a relative maximum value for the objective function, you can use one of the following Derivative Tests:

First Derivative Test

Assume f is continuous on an interval that contains a critical point c , and assume f is differentiable on an interval containing c , except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c , then f has a **local maximum** at c .
- If f' changes sign from negative to positive as x increases through c , then f has a **local minimum** at c .
- If f' is positive on both sides of c or negative on both sides of c , then f has no local extreme value at c .

Second Derivative Test

Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c .
- If $f''(c) < 0$, then f has a local maximum at c .
- If $f''(c) = 0$, then the test is inconclusive; f may have a local minimum, a local maximum, or neither at c .

These two tests identify where a function has a local extreme value, but in many cases we want to establish that the function has an absolute extreme value. If the objective function is defined on a closed interval, we can use the Extreme Value Theorem to determine if the absolute extreme occurs at a critical point within the interval, or at one of the endpoints. If the objective function is not defined on a closed interval, we can use the following theorem:

One Local Extremum Implies Absolute Extremum

Suppose f is continuous on an interval I that contains exactly one local extremum at c .

- If a local maximum occurs at c , then $f(c)$ is the absolute maximum of f on I .
- If a local minimum occurs at c , then $f(c)$ is the absolute minimum of f on I .

As you work through the optimization problems for today's activity, be sure to use the guidelines and theorems given to you above, keep your work organized and clear, and be sure to explain in each case why your answer is correct.

Example 1

An 8-ft-tall fence runs parallel to the wall of a house at a distance of 5 ft. Find the length of the shortest ladder that extends from the ground over the fence to the house, without touching the fence. Assume the vertical wall of the house is 20 ft high and the horizontal ground extends 20 ft from the fence.

Draw a picture of the situation. Label the ladder length L , label the distance that the ladder goes up the house with b , and label the distance from the base of the fence to the bottom of the ladder with x . What is an equation relating the variables L , b , and x ? Can you use similar triangles to come up with another equation that relates b and x ? Talk about the limitations on the possible values of x .

Example 2

Find the radius and height that minimize the amount of material used to make a pop can with a volume of 354 cm^3 , given that the can has double thickness in the top and bottom surfaces.

Draw a picture of the right circular cylinder that represents the pop can. use r to label the radius and h to label the height. Write down a formula involving r and h that gives the Volume of the pop can (which has to be 354 cm^3). Can you write down a formula for the amount of materials M we need to use (the surface area of the pop can) using the area of the sides, the area of the top of the can, and the area of the bottom of the can? (Hint: if you take the cylindrical side surface of the can and flatten it out, you get a rectangle. One edge length is the height h of the can, the other side length is the circumference of the circle.)

Example 3

A window consists of a rectangular pane of glass surmounted by a semicircular pane of glass. If the perimeter of the window is 20 ft., determine the dimensions of the window that maximize the total window area.

Draw a picture of the window. Label the height of the rectangular portion x , and the radius of the semicircle as r . Write a formula for the perimeter of the window in terms of x and r (which is equal to 20 ft.), and write a formula for the total area of the window in terms of x and r . Can you combine the two equations to get a formula for the area of the window that is only in terms of r , with x eliminated? Differentiate this function $A(r)$ with respect to r , and find the critical points.

Example 4 (to be done in the live session)

An island is 3.5 mi from the nearest point on a straight shoreline; that point is 8 mi from a power station. A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. Assume it costs \$2400/mi to lay underwater cable and \$1200/mi to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the total cost of the project?

PLTL Calculus 1 - Session 7

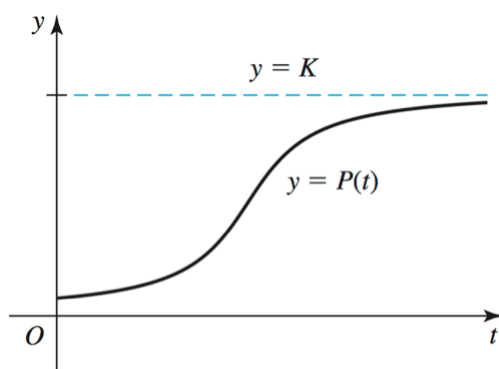
Before the live PLTL session on Monday, students should read the background material for each section and attempt the problems that are highlighted.

1. Logs and Exponentials

Scientists often use the following logistic growth function to model population growth:

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-r_0 t}}$$

where P_0 is the initial population at time $t = 0$, K is the carrying capacity (the theoretical upper bound on the total population that the surrounding environment can support), and r_0 is the base growth rate. This figure shows the sigmoid (S-shaped) curve associated with a typical logistic model.



Example 1

When a reservoir is created by a new dam, $P_0 = 50$ fish are introduced into the reservoir, which has an estimated carrying capacity of $K = 8000$ fish. The estimated initial growth rate is 50%, so $r_0 = 0.5$.

- (a) Write down a logistic growth model for this population of fish, and graph the function on your calculator in an appropriate window that produces the sigmoid curve on the screen. Explain your process for determining this window - how did you know how to set the min and max y -values so you can view the entire graph?

- (b) Determine how long it takes for the population to reach 5000 fish, and how fast the population is growing at this time. Then determine how long it takes to reach 90% of the carrying capacity, and how fast the population is growing at this time as well.

- (c) Use a graph of $P'(t)$ to determine when the population is growing the fastest.

Example 2

The graph of $y = x^{\ln x}$ has one horizontal tangent line. Find its equation.

2. Inverse Trig Derivatives

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \quad \frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

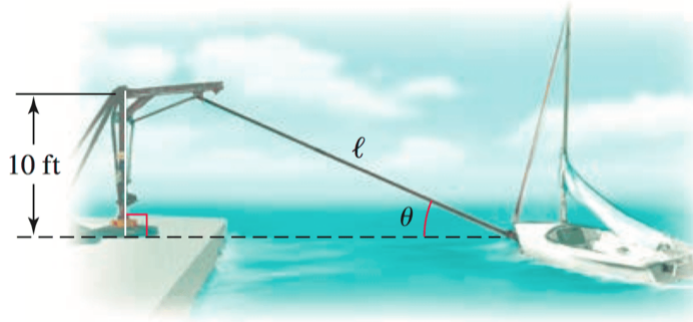
$$\frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}} \quad \frac{d}{dx} (\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2-1}}, \text{ for } |x| > 1$$

Example 3

Find an equation of the tangent line to $f(x) = \cos^{-1}(x^2)$ at the point where $x = \sqrt{2}/2$.
Don't forget to use the chain rule!

Example 4

A boat is towed toward a dock by a cable attached to a winch that stands 10 feet above the water level. Let θ be the angle of elevation of the winch and let ℓ be the length of the cable as the boat is towed to the dock.



- (a) Show that the rate of change of θ with respect to ℓ is

$$\frac{d\theta}{d\ell} = -\frac{10}{\ell\sqrt{\ell^2 - 100}}$$

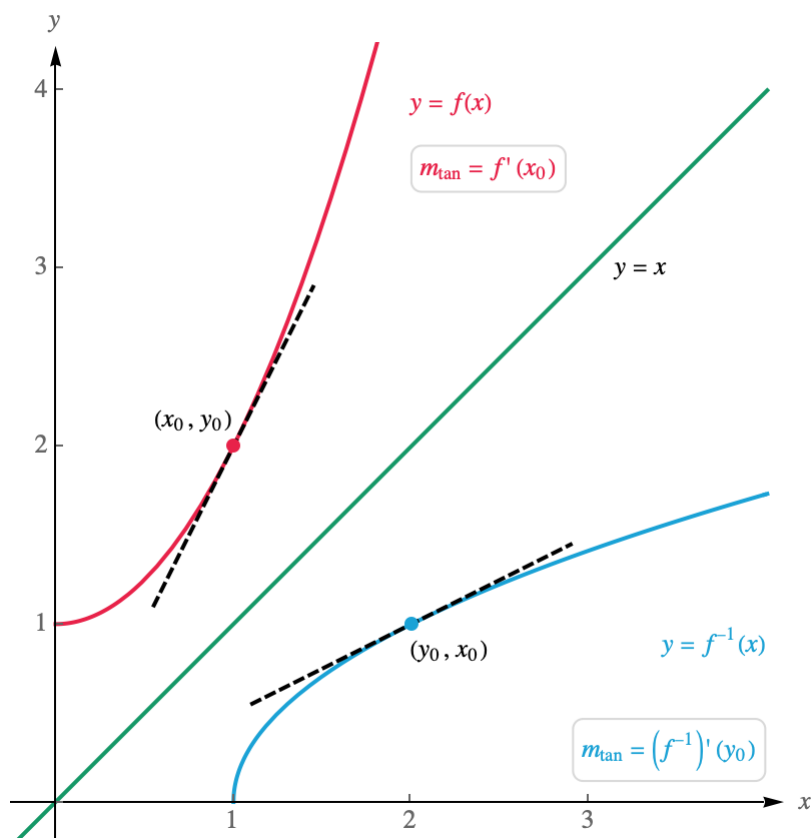
- (b) Calculate the following limit, and explain what happens as the last foot of cable is reeled in:

$$\lim_{\ell \rightarrow 10^+} \frac{d\theta}{d\ell}$$

- (c) From the picture, you can see that θ increases as the boat is towed to the dock. So, why is $d\theta/d\ell$ negative?

3. General Inverse Derivatives

We want to be able to calculate the rate of change of an inverse function without having to first find a formula for that inverse. If you look at this graph of a function $f(x)$ and its inverse $f^{-1}(x)$, you can see that inverse function graphs are reflections of each other across the line $y = x$ because for an inverse we are switching the roles of x and y .



Notice the connection between the slopes at the inverse points (x_0, y_0) on $f(x)$ and (y_0, x_0) on $f^{-1}(x)$: they are reciprocals.

Here's how to find the slope of an inverse function at a point without finding a formula for the inverse itself.

Let f be differentiable and have an inverse on an interval I . If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \text{where } y_0 = f(x_0).$$

Example 5

Use the table to determine the indicated derivatives, or state that they cannot be determined with the given information.

x	-2	-1	0	1	2
$f(x)$	2	3	4	6	7
$f'(x)$	1	$1/2$	2	$3/2$	1

(a) $(f^{-1})'(4)$

(c) $(f^{-1})'(6)$

(b) $(f^{-1})'(1)$

(d) $(f^{-1})'(2)$

Example 6

If f is a one-to-one function with $f(3) = 8$ and $f'(3) = 7$, find the equation of the tangent line to $y = f^{-1}(x)$ at $x = 8$.

Math 125 PLTL 11: L'Hôpital's Rule

Remember that if $f(x)$ is continuous at $x = a$, then:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Example: $\lim_{x \rightarrow 3} x^2$

Sometimes we don't even need continuity to determine a limit:

Example: $\lim_{x \rightarrow 0^+} \frac{1}{x}$

Other times, we get stuck on what's called an "Indeterminate Form":

Definition - Indeterminate Form

An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.

Example: $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

Form: $0/0$

Is this 0, since we approach 0 in the numerator?

OR: is this ∞ , since we approach 0 in the denominator?

Since it isn't clear, we call this form INDETERMINATE.

L'Hôpital's Rule is a method for dealing with certain indeterminate form limits.

L'Hôpital's Rule

Suppose f and g are differentiable functions on an open interval I containing the point $x = a$, with $g'(x) \neq 0$ on I when $x \neq a$.

If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $-\frac{\infty}{\infty}$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists as a real number, ∞ , or $-\infty$.

1. L'Hôpital's Rule is still valid if $x \rightarrow a$ is replaced by any of $x \rightarrow a^+$, $x \rightarrow a^-$, $x \rightarrow \infty$, or $x \rightarrow -\infty$. In the last two of these cases, there must be a greatest x -value beyond which both f and g are differentiable at every point.
2. If the limit on the right does not exist because it is not a real number, ∞ , or $-\infty$, this DOES NOT mean that the original limit also does not exist.

Back to our example: $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x}$

Example Evaluate $\lim_{x \rightarrow 0^+} \frac{1 - \ln(x)}{1 + \ln(x)}$.

Example Evaluate $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{5x^2}$.

L'Hôpital's Rule only applies directly to the indeterminate forms

$$\frac{0}{0}, \frac{\infty}{\infty}, \text{ or } -\frac{\infty}{\infty}$$

There are many other indeterminate forms; the idea will be to use a combination of algebra techniques and/or limit laws to express them as one of the forms eligible for L'Hôpital's Rule.

One such indeterminate form is $0 \cdot \infty$. We are going to use the facts that, as limits, $1/\infty = 0$ and $1/0 = \infty$ to rewrite this indeterminate form as one that will work with L'Hôpital's Rule.

Example Evaluate $\lim_{x \rightarrow 0^+} x \ln(x)$

Example Evaluate $\lim_{x \rightarrow 1^-} (1 - x) \tan\left(\frac{\pi x}{2}\right)$.

Now we are going to look at exponential indeterminate forms. We can walk through the first example together, then state what to do in general.

Example $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x} \right)^x$

Indeterminate Form: 1^∞

L'Hôpital's Rule cannot be applied directly to this type of form.

We can use exp/log/limit properties to rewrite the expression, following these steps:

1. Natural logarithm and natural exponential functions are inverses
2. Power property for log arguments: $\ln(f(x)^{g(x)}) = g(x) \ln(f(x))$
3. Exponential functions are continuous

Method for evaluating limits of indeterminate form 1^∞ , 0^0 , or ∞^0

Assume that $L = \lim_{x \rightarrow a} (x)^{g(x)}$ has one of these indeterminate forms.

1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

$$L = \lim_{x \rightarrow a} e^{\ln(f(x)g(x))}$$

2. Use the power property of logarithm arguments to write

$$= \lim_{x \rightarrow a} e^{(x) \ln f(x)}$$

3. Use continuity of the exponential function to write

$$L = e^{\lim_{x \rightarrow a} (x) \ln f(x)}$$

4. Rewrite multiplication as division by the reciprocal:

$$L = e^{\lim_{x \rightarrow a} \left(\frac{\ln(f(x))}{1/g(x)} \right)}$$

5. Use L'Hôpital's Rule to evaluate this limit expression.

Example Use L'Hôpital's Rule to evaluate the limit as x goes to infinity of the function $x^{\tan x}$.