Mathematics Competition

Indiana University of Pennsylvania

2016

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not necessarily expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted. Calculators, phones, etc. are prohibited.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by a proctor to do so.

1. When simplified, the expression $\left(\dfrac{14a^2b^7c}{2a^5bc^3}\right)^{-2}$ is equal to:

A. $\dfrac{49b^{12}}{a^6c^4}$

B. $\dfrac{a^6c^4}{49b^{12}}$

C. $\dfrac{a^6c^4}{14b^{12}}$

D. $\dfrac{14b^{12}}{a^6c^4}$

E. $\dfrac{4a^{10}b^{12}c^6}{196a^4b^{14}c^2}$

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2. Consider the system of equations

$$x^2 - y^2 - 4x + 6y - 4 = 0\,,$$
$$x^2 + y^2 - 4x - 6y + 12 = 0\,.$$

The solution set of the system is:

A. $\{(4, 2)\}$

B. $\{(2, 2)\}$

C. $\{(4, 2)\,, (2, 2)\}$

D. $\{(2, 2)\,, (2, 4)\}$

E. None of these

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3. Two six-sided dice are labeled with the following numbers:

Die #1: 1–2–2–3–3–4
Die #2: 1–3–4–5–6–8

When both dice are rolled, the probability that their sum is 2 is:

A. 0

B. $\dfrac{1}{12}$

C. $\dfrac{1}{36}$

D. $\dfrac{2}{12}$

E. $\dfrac{2}{36}$

---

4. The area of a circle is 36π square units and the center of the circle is $(-2, 3)$. The point that does not lie on this circle is:

A. $(-2, -3)$

B. $(4, 3)$

C. $(-2, 9)$

D. $(-8, 3)$

E. None of these

---

5. A math test has 25 problems. Some problems are worth 2 points and some are worth 3 points. The test is worth 60 total points. The number of 3-point questions on the test is:

A. 10

B. 12

C. 13

D. 15

E. None of these

---

6. Compared to the graph of $y = f(x)$, the graph of $y = f(x-2) + 7$ is:

A. Shifted 2 units to the right and shifted 7 units up

B. Shifted 2 units to the left and shifted 7 units up

C. Horizontally stretched by a factor of 2 and shifted 7 units down

D. Horizontally stretched by a factor of 2 and shifted 7 units up

E. Shifted 7 units to the right and shifted 2 units down

---

7. Of the statements below, the one which is equivalent to $a\log_c(m) + b\log_c(n)$ is:

A. $\log_c\left((mn)^{ab}\right)$

B. $\log_c\left(m^a + n^b\right)$

C. $\log_c\left(m^a n^b\right)$

D. $\log_c(m^a)\log_c\left(n^b\right)$

E. $\log_c\left(\dfrac{m^a}{n^b}\right)$

---

8. The value of $\cos^{-1}\left(\cos\left(\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)\right)\right)$ is:

   A. $-\frac{3\pi}{4}$
   B. $-\frac{\pi}{4}$
   C. 0
   D. $\frac{\pi}{4}$
   E. $\frac{3\pi}{4}$

---

9. Complete the following division: $(3x^4 + 2x^3 + 4x^2 + x - 5) \div (x^2 + 2)$. The remainder is:

   A. $3x^2 + 2x - 2$
   B. $3x^2 - 4x - 12$
   C. $x^2 + 2$
   D. $x - 29$
   E. $-3x - 1$

---

10. Consider the circle in the *xy*-plane that has a center of $(1, -2)$ and passes through the point $(1, -5)$. The area of this circle is:

    A. 3π
    B. $3\sqrt{2}\pi$
    C. 6π
    D. 9π
    E. 18π

---

11. The set which *contains* all of the solutions to $|x - 1| + |x - 2| = 3$ is:

    A. $\{-5, -3, -1, 1, 3, 5, 7\}$
    B. $\{-6, -4, -2, 0, 2, 4, 6\}$
    C. $\{-3, -2, -1, 0, 1, 2, 3\}$
    D. $\{-5, -2, 1, 4, 6, 7\}$
    E. $\{-1, 2, 5, 9\}$

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12. A water tank in the shape of a right circular cylinder is 35 feet long and 10 feet in diameter. The amount of sheet metal used in the construction of this tank is:

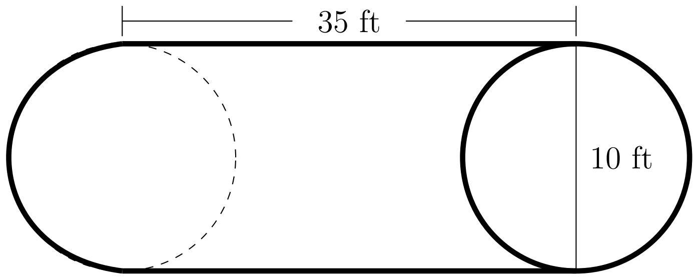
   A. 350π ft^2

   B. 400π ft^2

   C. 375π ft^2

   D. 60π ft^2

   E. 90π ft^2



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13. Let $\angle 1$ and $\angle 2$ be vertical angles where $m(\angle 1) = 5y + \frac{7}{3}$ and $m(\angle 2) = \frac{2}{3}x + \frac{1}{3}$. Also, $\angle 2$ is supplementary to $\angle 3$ and $m(\angle 3) = 10x - \frac{37}{3}$. The value of $x + y$ is:

   A. $\dfrac{167}{16}$

   B. $\dfrac{766}{15}$

   C. 180

   D. 20

   E. None of these

---

14. If $\sec(\theta) = -3$ with $\pi < \theta < \dfrac{3\pi}{2}$, then the value of $\cot(\theta)$ is:

   A. $\sqrt{3}$

   B. $\dfrac{\sqrt{2}}{2}$

   C. $\sqrt{2}$

   D. $\dfrac{\sqrt{2}}{4}$

   E. $2\sqrt{2}$

---

15. The third degree polynomial that has 2 and $1 - i$ as two of its zeros is:

   A. $x^3 - 4x^2 + 6x - 4$

   B. $x^3 + 4x^2 - 6x + 1$

   C. $x^3 + 6x^2 - 4x + 2$

   D. $x^3 + 2x^2 - x + 6$

   E. $2x^3 - 4x^2 + 6x - 4$

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16. A diagonal of an octagon is a segment that connects two non-adjacent vertices. The maximum number of possible diagonals in an octagon is:

A. 36

B. 20

C. 56

D. 28

E. None of these

---

17. The value of x in the equation $\log_5(x+2) - \log_5(x) = 2$ is:

A. 1

B. $\frac{1}{4}$

C. $\frac{1}{12}$

D. $\frac{1}{24}$

E. None of these

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18. When the polynomial $80z^{2m+1} + 64z^{m+1} - 84z$ is factored completely, the sum of the factors is:

A. $4z + 12z^m - 4$

B. $144z^{m+1} - 84z$

C. $4z^m + 12z^{2m} - 4$

D. $12z^m - 4$

E. $4z + 12z^m + 4$

---

19. The value of x in $e^{2x} - 4e^x = 5$ is:

A. $\ln(1)$

B. $\ln(4)$

C. $\ln(5)$

D. 1

E. None of these

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20. The amount of water that must be evaporated from 30 liters of a 40% salt solution to obtain a 60% salt solution is:

A. 12 liters

B. 18 liters

C. 20 liters

D. 60 liters

E. 10 liters

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21. A square is inscribed in a circle. If the ratio of circumference to area for the circle is 16, the ratio of perimeter to area for the square is:

A. $32\sqrt{2}$

B. $2\sqrt{16}$

C. $\frac{1}{8}$

D. $\frac{\sqrt{16}}{8}$

E. $16\sqrt{2}$

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22. Let a, b, and c be the x-intercepts of $y = 2x^3 + 7x^2 - 14x - 40$, with $a < b < c$. Then the value of $a - 3b + 2c$ is:

A. 7

B. -20

C. $-7/2$

D. 20

E. $-3/2$

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23. A wooden sign is made of three slats of the same size, shown by the shaded regions in the figure below. The entire sign occupies a total area of 4.5 square feet. The length of the sign measures twice the height. There is a 1.5 inch space between each slat. The area of each slat is:

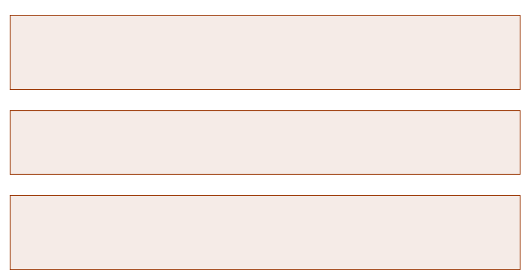
A. 15 ft^2

B. 9 ft^2

C. 1.25 ft^2

D. 1.5 ft^2

E. 2 ft^2



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24. The interval that contains the real solution to $\sqrt{x} + \sqrt[3]{x} = 36$ is:

A. $[1, 6]$
B. $[8, 13]$
C. $[45, 54]$
D. $[92, 100]$
E. None of these

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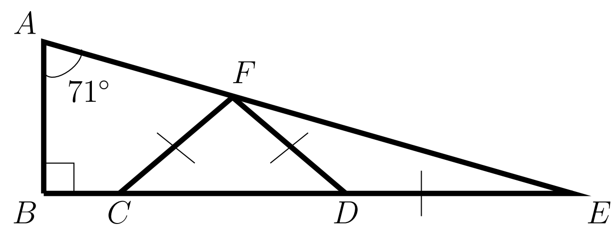
25. Given that the measure of $\angle BAF$ is 71°, then the measure of $\angle FCB$ is:

A. 161°
B. 109°
C. 99.5°
D. 142°
E. None of these



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26. The shortest distance between the point $(-2, 2)$ and the circle $x^2 + y^2 - 12x + 8y + 43 = 0$ is:

A. 3
B. 10
C. 7
D. $\sqrt{43}$
E. None of these

---

27. Consider the quadratic function $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers with $ac \neq 0$. Suppose that r is a real number such that $f(r) = 0$. A linear function $g(x)$ satisfying the equation $r \cdot g(r) = 1$ is:

A. $g(x) = \frac{a}{c}x + \frac{b}{c}$

B. $g(x) = -\frac{a}{c}x - \frac{b}{c}$

C. $g(x) = -\frac{a}{c}x + \frac{b}{c}$

D. $g(x) = \frac{a}{c}x - \frac{b}{c}$

E. None of these

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28. The sum of the solutions to $\cos^2(x) = \cos(x)$ on the interval $[0, 2\pi)$ is:

A. 2π

B. π

C. $\frac{5\pi}{2}$

D. $\frac{7\pi}{4}$

E. None of these

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29. In a game show, exactly one of three closed boxes contains the keys to a brand new car. As the first place contestant, you are given the opportunity to select one of the three boxes in an attempt to win the new car should the keys be in the box you chose. However, before viewing the contents of the box you selected, the game show host opens one of the other two boxes and demonstrates to you that the keys to the new car are not in that box. You have the option to either stay with your initial selection or to switch to the remaining unopened box. The probability of selecting the box containing the keys to the new car if you choose to switch is:

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. 0

D. 1

E. None of these

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30. A regular dodecahedron is a polyhedron comprised of 12 regular pentagonal faces. The apothem is a perpendicular line segment drawn from the center of a pentagon to the midpoint of an edge. Suppose each pentagonal face has an apothem of length a and sides of length $2b$. The surface area of the dodecahedron is:

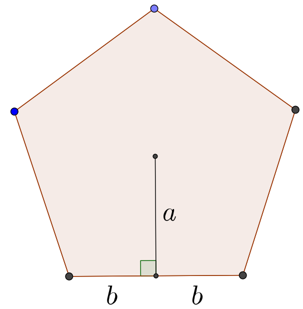
A. $10\sqrt{3}ab$

B. $\frac{37}{2}ab$

C. $60ab$

D. $\frac{1037}{13}ab$

E. None of these



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31. One thousand different students are divided into three math courses – Algebra, Statistics, and Trigonometry. Thirty-five percent of the 1000 students are juniors. In particular, fifty percent of all students who take Algebra are juniors, twenty percent of all students who take Statistics are juniors, and thirty percent of all students who take Trigonometry are juniors. The total number of Algebra students is 10 more than half of the Trigonometry and Statistics students combined. How many juniors are in each math class?

A. 340 in Algebra, 180 in Statistics, 480 in Trigonometry

B. 175 in Algebra, 70 in Statistics, 105 in Trigonometry

C. 500 in Algebra, 200 in Statistics, 300 in Trigonometry

D. 170 in Algebra, 70 in Statistics, 110 in Trigonometry

E. None of these

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32. Chris stands on the edge of a building at a height of 77 feet and throws a ball upward with an initial velocity of 16 feet per second. The ball eventually falls all the way to the ground. The height of the ball after t seconds is modeled by the function $h(t) = -16t^2 + 16t + 77$. The time it will take for the ball to reach the ground is:

A. 0.75 seconds

B. 1.75 seconds

C. 2.75 seconds

D. 3.75 seconds

E. None of these

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33. The notation

$$\mu = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{pmatrix}$$

represents the function $\mu(x_1) = y_1$, $\mu(x_2) = y_2$, etc. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}.$$

The inverse function of the composite function $\sigma \circ \tau$ is the function ω, where:

A. $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$

B. $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$

C. $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$

D. $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix}$

E. None of these

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34. In the figure below, $\overline{CD}$ is perpendicular to $\overline{AB}$ and $\frac{AD}{CD} = \frac{CD}{DB}$. The measure of the angle $\angle ACB$ is:

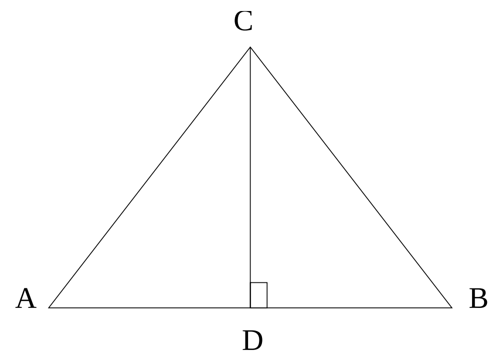
A. 60°

B. 75°

C. 45°

D. 90°

E. None of these



35. A city's roadways are arranged in a rectangular grid, see below. Pedestrian crosswalks are one-way. They are designed to only permit you to move North or East. The number of distinct walking paths from point A to point B is:

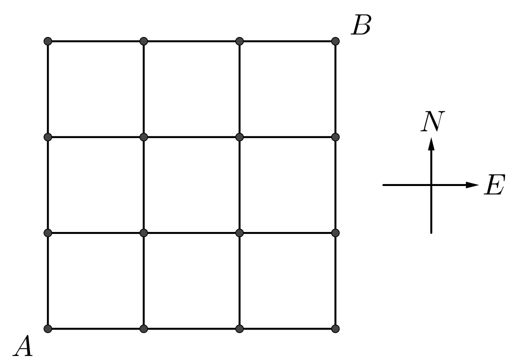
A. 16

B. 18

C. 19

D. 22

E. None of these



36. Decibels are used to quantify losses associated with atmospheric interference in a communication system. The ratio of the power (watts) received to the power transmitted (watts) is often compared. Often, *watts* are transmitted, but losses due to the atmosphere typically correspond to *milliwatts* being received:

$$\text{dB} = 10 \log \left(\frac{\text{Power received}}{\text{Power transmitted}} \right)$$

If 1 W of power is transmitted and 1 mW is received, the power loss in dB is:

A. 0.3 dB

B. 3 dB

C. 30 dB

D. 300 dB

E. None of these

37. Suppose three runners - Larry, Moe, and Curly - each run at their own constant rate during the entirety of a 200 meter race. If Larry finishes 20 meters ahead of Moe, and Moe finishes 20 meters ahead of Curly, how many meters ahead of Curly did Larry finish?

A. 40 meters

B. 38 meters

C. 36 meters

D. 34 meters

E. 32 meters

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38. Max has four different colored marbles that are the same size in a bag. He randomly chooses one marble, looks at it, and places it back into the bag. He repeats this process three more times. The probability that he has seen at most three different colored marbles is:

A. $\frac{1}{64}$

B. $\frac{3}{32}$

C. $\frac{7}{8}$

D. $\frac{29}{32}$

E. $\frac{1}{128}$

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39. In the infinite product $e^a \cdot e^{a/2} \cdot e^{a/4} \cdot e^{a/8} \cdot \ldots = e^8$, the value of a is:

A. 1

B. 4

C. 8

D. 16

E. None of these

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40. If x and y are real numbers with $0 < x < \frac{\pi}{2}$ and $\sin(x) = y$, the the value of $\sin(2x)$ is:

A. $2y$

B. y^2

C. $y\sqrt{1-y^2}$

D. $-y\sqrt{1-y^2}$

E. $2y\sqrt{1-y^2}$

---

41. Consider the following expression:

$$\cfrac{3}{4+\cfrac{3}{4+\cfrac{3}{4+\cdots}}}$$

When this pattern is continued infinitely, the expression simplifies to:

A. $2+\sqrt{7}$

B. $2-\sqrt{7}$

C. 7

D. 3.4

E. None of these

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42. The number of solutions to $8\sin^2(x)\cos^2(x)+3=4\sin^2(x)+6\cos^2(x)$ on the interval $(-\pi,\pi)$ is:

A. 2

B. 4

C. 8

D. 10

E. 16

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43. The solution set to $\dfrac{x^2-4x}{x+1}+\dfrac{x+3}{x+2}\geq\dfrac{x+6}{x^2+3x+2}$ is:

A. $(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$

B. $(-\infty,-2)\cup[4,\infty)$

C. $[-5,-2)\cup(3,\infty)$

D. $(-2,-1)\cup[3,\infty)$

E. None of these

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44. A stack of six cards is labeled 1 through 6 and placed in increasing order on a table. The cards are shuffled as follows:

- The top three cards are moved as a group to a separate pile on the right.
- The cards are restacked by first taking a card from the left pile and then placing a card from the right pile on top. This is done repeatedly until a single stack of cards is formed.

The number of these shuffles that will produce the cards in their original increasing order is:

A. 1

B. 3

C. 4

D. 6

E. 10

---

45. A regular hexagon with sides of length a is tiled in the plane, see below. Let H be the area enclosed by the hexagons tiled in the plane. Let C be the area of the circumscribing circle. The ratio H/C is:

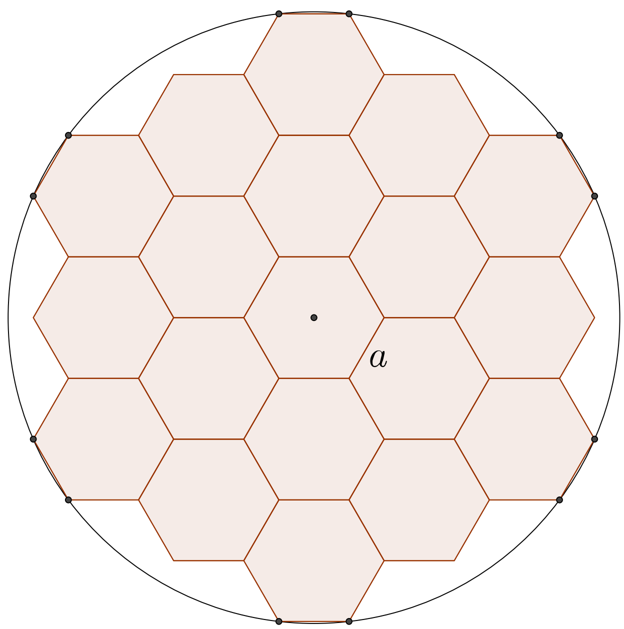
A. $\frac{\sqrt{5}}{\pi}$

B. $\frac{3}{\pi}$

C. $\frac{10\sqrt{2}}{\pi}$

D. $\frac{3\sqrt{3}}{2\pi}$

E. None of these



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46. The diagram shows the top of a table that is made up of a rectangular center with an isosceles triangle on either side. The triangles are the same size. The measurement down the center of the table, from point A to point B, is 8 feet. The measurement across the table, from point C to point D, is 4 feet. The measurement from point A to point D is 26 inches. The total area of the top of the table is:

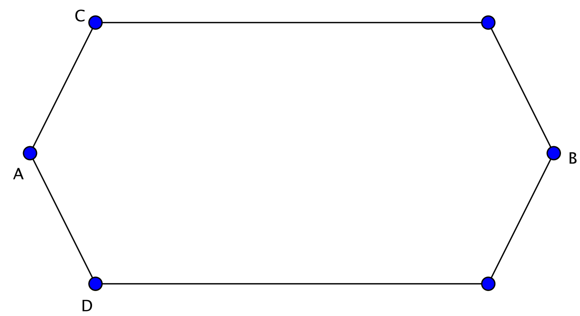
A. 344 ft^2

B. 384 ft^2

C. 4128 in^2

D. 4608 in^2

E. 5088 in^2



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47. Ten light bulbs (each a different color) are arranged in a circular fashion. Suppose that exactly three out of the ten light bulbs are turned on. The number of ways that three light bulbs are turned on such that no two adjacent light bulbs are turned on is equal to:

A. 50

B. 100

C. 70

D. 120

E. None of these

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48. The mineral calcite forms crystals with a variety of shapes. One particular growth pattern is rhombohedral, see below. A rhombohedron is a solid with edges of equal length a and three unique pairs of faces: front-back, right-left, and top-bottom. If $a = 1$, $\alpha = \pi/3, \beta = \pi/3$, and $\gamma = \pi/4$, then the length of the diagonal $\overline{AR}$ is:

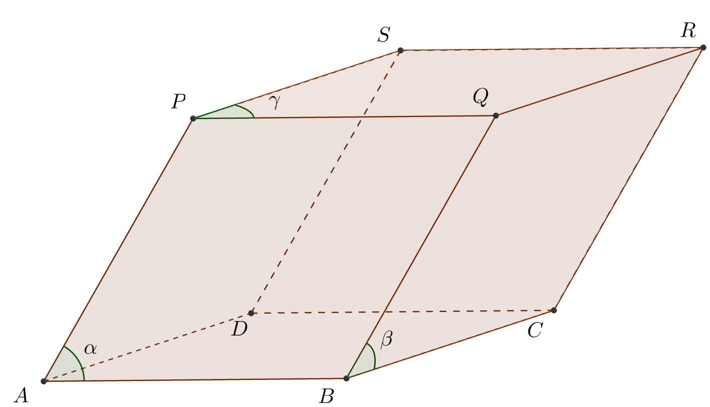
A. $\sqrt{3+\sqrt{2}} + \sqrt{2}$

B. $\sqrt{3+\sqrt{2}} + \sqrt{3} + \frac{1}{2}$

C. $\sqrt{3+\sqrt{2}}$

D. $\sqrt{5+\sqrt{2}}$

E. None of these



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49. A spider and a fly are located on a cylinder at points $S(1,1,2)$ and $F(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2}, 2+\sqrt{5})$. The spider is crawling around the cylinder to catch the fly. The length of the shortest path the spider may take to catch the fly is:

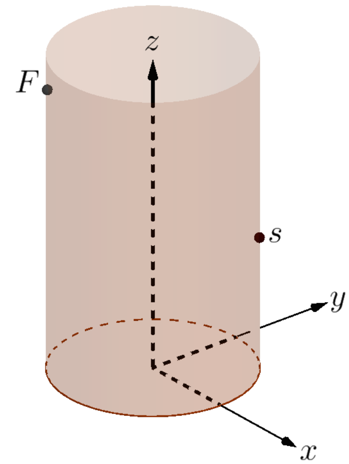
A. $\sqrt{9+\sqrt{2}+\sqrt{6}}$

B. $= \sqrt{5+\frac{121}{72}\pi^2}$

C. $\sqrt{11}\pi$

D. $\sqrt{23}\pi$

E. None of these



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50. For any given function $f(x)$, define $C_n\big(f(x)\big)$ to be the n-fold composite function of $f(x)$ with itself. Thus,

$$C_1\big(f(x)\big) = f(x),$$
$$C_2\big(f(x)\big) = \big(f \circ f\big)(x) = f\big(f(x)\big),$$
$$C_3\big(f(x)\big) = \big(f \circ f \circ f\big)(x) = f\Big(f\big(f(x)\big)\Big).$$

For a fixed real number a, define $f_a(x) = a + x + ax$. If $C_n\big(f_a(x)\big) = px + q$, then the value of q/p is:

A. $1 - \dfrac{1}{(a+1)^n}$

B. $\dfrac{a}{a+1}$

C. $1 + \dfrac{1}{a}$

D. $\dfrac{1}{a+1}$

E. None of these

Answer Key
2016 IUP High School Mathematics Competition

1. B
2. D
3. C
4. E
5. A
6. A
7. C
8. D
9. E
10. D
11. C
12. B
13. D
14. D
15. A
16. B
17. C
18. A
19. C
20. E
21. E
22. A
23. C
24. E
25. D
26. C
27. B
28. A
29. E
30. C
31. E
32. C
33. D
34. D
35. E
36. C
37. B
38. D
39. B
40. E
41. E
42. C
43. D
44. D
45. D
46. C
47. A
48. D
49. B
50. A