Mathematics Competition

Indiana University of Pennsylvania

2010

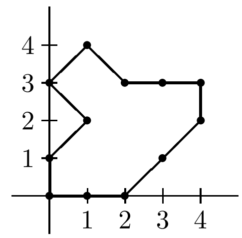
DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The area of the polygonal region shown is:

(A) 4
(B) 6
(C) 8
(D) 10
(E) 12



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2. The solution set to the equation

$$x^2(x-1) - 4(x-1) = 0$$

is:

(A) $\{2, -2\}$ (B) $\{1, -1\}$ (C) $\{1\}$ (D) $\{0, 1\}$ (E) $\{1, 2, -2\}$

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3. The number of 5-digit ZIP codes that are possible if 0 cannot be used as the first digit is:

(A) 90,000 (B) 100,000 (C) 9,000 (D) 10,000 (E) 99,000

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4. The values of b such that $-2x^2 + bx + 1 = 0$ has exactly two real solutions are given by the set:

(A) $\{b : b \text{ is any real number}\}$ (B) $\{b : b \geq 0\}$ (C) $\{b : b < 0\}$
(D) $\{b : -1 \leq b \leq 1\}$ (E) $\emptyset$

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5. The expression

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta}$$

is equal to:

(A) $2 + 2\cos\theta$ (B) 1 (C) $(2 + 2\cos\theta)\csc\theta$ (D) $2\csc\theta$
(E) none of these

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6. The expression $\sqrt{x} - \left(-\sqrt{4x}\right)$ is equal to:

(A) $2x$ (B) $\sqrt{5x}$ (C) $x\sqrt{5}$ (D) $3\sqrt{x}$ (E) none of these

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7. The measures of the angles in the triangle ABC for which $\angle A = (40x - 5)^\circ$, $\angle B = (22 - 2x)^\circ$, and $\angle C = (7 + x)^\circ$ for some x are:

(A) $\angle A = 155^\circ, \angle B = 18^\circ, \angle C = 7^\circ$
(B) $\angle A = 155^\circ, \angle B = 14^\circ, \angle C = 11^\circ$
(C) $\angle A = 25^\circ, \angle B = 22^\circ, \angle C = 3^\circ$
(D) $\angle A = 25^\circ, \angle B = 10^\circ, \angle C = 15^\circ$
(E) none of these

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8. The solution set to the equation

$$x^{3/4} - 4x^{1/4} = 0$$

is:

(A) $\{0\}$ (B) $\{0, 16\}$ (C) $\{4, 16\}$ (D) $\{0, 4, 16\}$ (E) none of these

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9. The only false statement among the following is:

(A) $\sin^{-1}(1/2) = \pi/6$
(B) $\tan^{-1}(-1) = -\pi/4$
(C) $\tan(\sin^{-1}(1/4)) = \sqrt{15}/15$
(D) $\cos(\cos^{-1}(1/3)) = (1/3)$
(E) $\sin(\cos^{-1}(x)) = -\sqrt{1 - x^2}$

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10. A square is picked up off a table. The number of different ways that it can be placed back on the table so that it exactly covers its original position is:

(A) 4 (B) 8 (C) 12 (D) 16 (E) 24

---

11. The set of all x-intercepts of the graph of $y = x^2 + x + 1$ is:

(A) $\{(0, 1)\}$
(B) $\{(-1/2, 3/4)\}$
(C) $\{(1/2, 0), (-1/2, 0)\}$
(D) $\{(3/4, 0), (-1/2, 0)\}$
(E) $\emptyset$

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12. Cindy wants to secure sequins on a piece of felt shaped like a trapezoid with a height of 8 cm and bases 16 cm and 10 cm. If sequins cost \$1.50 per cm^2 of coverage, then the cost of the sequins she needs is:

(A) \$156 (B) \$63 (C) \$51 (D) \$180 (E) none of these

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13. Suppose you are told that a student's second test score was eight points higher than his first test score. His third test score was an 88. If he has a B average (between 80 and 89 inclusive), then the possible scores on his first test are those scores in the interval:

(A) $[50, 75]$ (B) $[65.5, 76]$ (C) $[70, 82]$ (D) $[72, 85.5]$ (E) $[87, 100]$

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14. Using the sum of an infinite geometric sequence, one can express the repeating decimal $0.121212\ldots$ as the fraction:

(A) $4/33$ (B) $2/15$ (C) $2/33$ (D) $40/333$ (E) $4/15$

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15. The number of rational solutions to the equation

$$|x^2 - 3x| = 2$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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16. To determine the height of a radio transmission tower, a surveyor walks off a distance of 300 meters from the base of the tower. The angle of elevation is measured and found to be $30°$. If the transit (the device used to measure the angle of elevation) is 2 meters off the ground when the sighting is taken, the height of the tower in meters is:

(A) $\dfrac{\tan 30°}{300} + 2$ (B) $\dfrac{300}{\tan 30°} + 2$ (C) $302 \tan 30°$ (D) $2 \tan 30° + 300$

(E) $300 \tan 30° + 2$

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17. Let $(5, 7)$ be a point on the graph of $f(x)$, and let $g(x) = f(x^2 - 4)$. Then a point that must be on the graph of $g(x)$ is:

(A) $(21, 45)$ (B) $(-3, 7)$ (C) $(5, 21)$ (D) $(-7, 45)$ (E) none of these

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18. If the points $(-12, -17)$, $(-3, k)$, and $(18, 53)$ are collinear, then the value of k is:

(A) 37 (B) 18 (C) $-2/7$ (D) $34/5$ (E) 4

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19. The expression

$$(\log_2 3)(\log_9 16)(\log_{64} 125)(\log_{625} 1296)(\log_{7776} 16807)(\log_{117649} 262144)$$

is equal to:

(A) 1 (B) 3 (C) $\ln 8^6$ (D) $\dfrac{\ln 9^7}{\ln 8^7}$ (E) $\log_2 262144$

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20. The solution set to the equation

$$9 + \frac{3}{x} - \frac{2}{x^2} = 0$$

is:

(A) $\{-2/3\}$ (B) $\{\pm 2/3, \pm 1/3\}$ (C) $\{-1/3, 0\}$ (D) $\{-2/3, 1/3\}$
(E) $\emptyset$

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21. Let A, B, and C be the points in $\mathbb{R}^3$ such that $A = (5, 8, -3)$, $B = (15, 8, 0)$, and $C = (0, 12, 5)$. The triangle ABC is:

(A) acute (B) right
(C) obtuse (D) nonexistent, since A, B, and C are not coplanar
(E) none of these

---

22. The number of two digit numbers that are divisible by 2 or by 5 is:

(A) 63 (B) 54 (C) 45 (D) 30 (E) 18

---

23. The solution set, in interval notation, to the inequality

$$\frac{(x-1)^2(x-2)}{x(x^2+2x+4)} \geq 0$$

is:

(A) $(-\infty, 0) \cup [1, 1] \cup [2, \infty)$ (B) $(-\infty, 0] \cup [1, 2]$ (C) $[0, 1] \cup (2, \infty)$
(D) $(0, 2]$ (E) $(0, 1] \cup [2, \infty)$

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24. A pyramid with a rectangular base has a volume of 30 m^3. Suppose the height is $x - 2$ and the dimensions of the base are $3x$ by $x + 5$. Then the value of x is:

(A) 3 m (B) $\sqrt{10}$ m (C) $2\sqrt{2}$ m (D) $2\sqrt{3}$ m (E) none of these

---

25. The number of liters of an 8% acid solution that must be mixed with 5 liters of a 2% acid solution and 4 liters of a 5% acid solution to obtain a solution that is 7% acid is:

(A) 3 (B) 13 (C) 23 (D) 33 (E) 43

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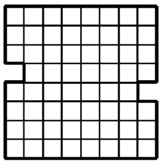
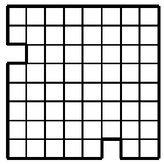
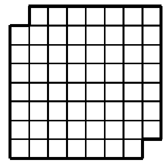
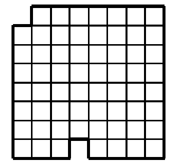
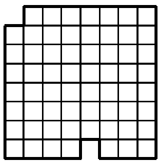
26. An example of a grid that can be completely covered by dominoes (2×1 rectangles) placed horizontally and/or vertically without any part of the domino going off of the grid is:

(A)  (B)  (C)  (D) 

(E) 

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27. If it is known that $\sin\alpha = 4/5$ with $\pi/2 < \alpha < \pi$ and that $\sin\beta = -2/\sqrt{5}$ with $\pi < \beta < 3\pi/2$, then the exact value of $\cos(\alpha+\beta)$ is:

(A) $\dfrac{11\sqrt{5}}{25}$ (B) $\dfrac{\sqrt{5}}{5}$ (C) $\dfrac{-8}{5\sqrt{5}}$ (D) $\dfrac{4\sqrt{5}-10}{5\sqrt{5}}$ (E) none of these

---

28. Four of the following points lie on a circle. The point that does not is:

(A) $(2,5)$ (B) $(6,1)$ (C) $(2,1)$ (D) $(5,2)$ (E) $(6,5)$

---

29. The solution set to the equation $x^{\ln x} = xe^2$ is:

(A) $\{2\}$ (B) $\{e^2\}$ (C) $\{e^2, e^{-2}\}$ (D) $\left\{e^{e^2}\right\}$ (E) none of these

---

30. The solution set to the sytem

$$\begin{aligned} 6x + 3y &= -3 \\ -4x - 2y &= -2 \end{aligned}$$

is:

(A) $\{(x, -2x-1) \mid x \text{ is a real number}\}$ (B) $\{(0,-1)\}$ (C) $\{(0,1)\}$
(D) $\{(0,-1),(0,1)\}$ (E) $\emptyset$

---

31. Let $ABCD$ be a quadrilateral with $AB = 3$, $BC = 4$, $CD = 12$, and $DA = 13$. If $\angle ABC$ is a right angle, then the area of $ABCD$ is:

(A) 36 (B) 30 (C) 25.5 (D) impossible to determine
(E) none of these

---

32. The geometric mean of 8, 32, 256, 8, and 2 is:

(A) 8 (B) 32 (C) 16 (D) 61.2 (E) none of these

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33. The number of real solutions to the equation

$$\sqrt{x-3}+\sqrt{2x+1}=4$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

34. The number of solutions to the equation $\cos 2x = \cos x$ with $0 \le x < 2\pi$ is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) none of these

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35. The value of $\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}$ is:

(A) 2 (B) $\dfrac{1+\sqrt{13}}{2}$ (C) $\dfrac{1+\sqrt{17}}{2}$ (D) $\dfrac{1+\sqrt{21}}{2}$ (E) 3

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36. James has two mixing bowls that are both perfect hemispheres. He places the smaller of the two on a table and then places the second bowl upside-down on top of it. The entire rim of the smaller bowl touches the inside of the large bowl and the rim of the large bowl rests on the table. The ratio of the diameter of the large bowl to the small bowl is:

(A) $\sqrt{2}$ (B) $\pi/2$ (C) $\sqrt{3}$ (D) $3/2$ (E) none of these

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37. You are painting a cube. Each face must be one solid color, either black or white. Two paintings are considered the same if rotating one of the cubes makes them appear identical. The number of distinctly different ways that the cube can be painted is:

(A) 6 (B) 7 (C) 10 (D) 16 (E) 64

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38. The number of real solutions to the equation

$$2^x+x-1=0$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

39. The value of $\cos \pi/32$ is:

(A) $\dfrac{\sqrt{2}}{16}$ (B) $\dfrac{\sqrt{2}}{2}$ (C) $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ (D) $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

(E) $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$

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40. The equation

$$\frac{x+2}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

is an identity for:

(A) $A = 2$, $B = -2$, $C = 3$ (B) $A = -2$, $B = 2$, $C = 3$
(C) $A = 1$, $B = -1$, $C = 0$ (D) $A = 2$, $B = 4$, $C = -3$
(E) no values of A, B, C

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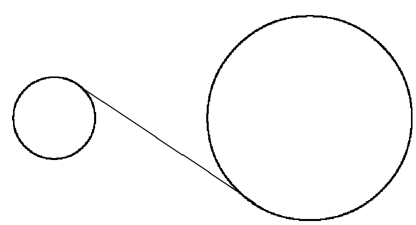
41. Two circles have radii of 8 and 20 respectively. The distance between their centers is 49. The length of the common tangent shown in the figure is:



(A) $3\sqrt{65}$
(B) $5\sqrt{42}$
(C) $21\sqrt{5}$
(D) $7\sqrt{33}$
(E) $14\sqrt{11}$

---

42. If $x > 0$ satisfies

$$\log_{2010} x + \log_x 2010 = 10,$$

then $\log_{2010} x$ is equal to:

(A) $5\log_{10} 2010 \pm \log_{10} 2010$ (B) 5 ± 1 (C) $5 \pm 2\sqrt{6}$ (D) 201
(E) none of these

---

43. The solution set to the system

$$\begin{aligned} 3xy - 2y^2 &= -2 \\ 9x^2 + 4y^2 &= 10 \end{aligned}$$

is:

(A) $\{(0,1)\}$ (B) $\{(\pm\sqrt{10}/3, 0)\}$ (C) $\{(\pm 1, \mp 1/2), (\pm\sqrt{2}/3, \pm\sqrt{2})\}$
(D) $\{\pm 1, \mp 1/2)\}$ (E) $\{(0,1), (\pm\sqrt{10}/3, 0)\}$

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44. A partially colorblind man misidentifies blue and green 20% of the time. There is a jar with 100 marbles—85 blue, 15 green. He draws one marble at random, looks at it, and says, "It is green." The probability that the marble is blue is:

(A) 1/5 (B) 4/5 (C) 85/100 (D) 17/29 (E) none of these

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45. The distance from the point $(3, 1)$ to the reflection of the line $y = 2x + 5$ across the line $y = -1 - x$ is:

(A) 2 (B) $2\sqrt{2}$ (C) $\sqrt{5}$ (D) $3\sqrt{5}$ (E) $3\sqrt{3}$

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46. Given $p(x) = x^4 - x^3 - 4x^2 + 3x - 1$, the 5-tuple of real numbers $(a_0, a_1, a_2, a_3, a_4)$ such that

$$p(x) = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4$$

is:

(A) $(-3, 7, 14, 7, -1)$ (B) $(6, 23, 18, 7, 1)$ (C) $(-3, 7, 14, 7, 1)$
(D) $(3, -7, -14, 7, 1)$ (E) none of these

---

47. The solution set to the equation

$$\log_3 9x = 3 \log_x 3$$

is:

(A) $\{3\}$ (B) $\{1\}$ (C) $\{1, 3\}$ (D) $\{9\}$ (E) none of these

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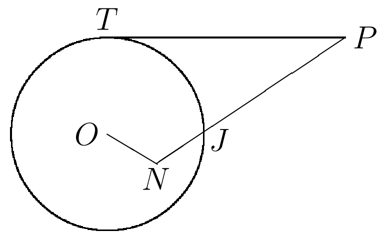
48. In the figure, $\overline{PT}$ is tangent to the circle centered at O and $\overline{PN}$ intersects this circle at J. If $PT = 21$, $PJ = 15$, and $ON = JN = 5$, then the radius of the circle is:



(A) $5\sqrt{3}$
(B) $5\sqrt{2}$
(C) $6\sqrt{2}$
(D) $3\sqrt{5}$
(E) $4\sqrt{3}$

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49. The sum of the distinct real solutions to the equation

$$x^4 + x^3 - 15x^2 + 16x + 4 = 0$$

is

(A) -3 (B) -1 (C) 1 (D) 2 (E) 7

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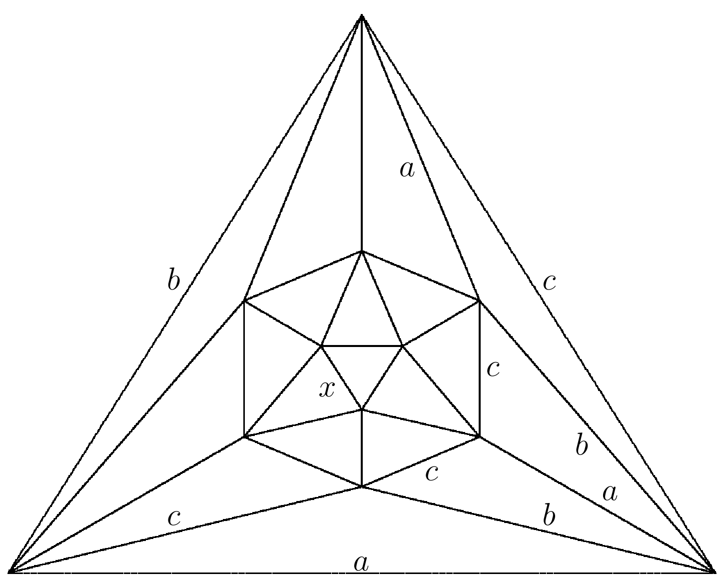
50. Suppose that weights a, b, and c are assigned to the edges of the graph as shown. If we continue assigning weights so that each triangle has one edge of each weight, then the set of all possible weights for the edge marked with an x is:



(A) $\{a\}$ (B) $\{b\}$ (C) $\{c\}$ (D) $\{a, b\}$ (E) $\{b, c\}$

Answer Key

1. D
2. E
3. A
4. A
5. D
6. D
7. B
8. B
9. E
10. B
11. E
12. A
13. D
14. A
15. C
16. E
17. B
18. E
19. B
20. D
21. C
22. B
23. A
24. B
25. D
26. E
27. A
28. D
29. E
30. E
31. A
32. C
33. B
34. C
35. B
36. A
37. C
38. B
39. E
40. A
41. D
42. C
43. C
44. D
45. C
46. C
47. E
48. C
49. A
50. D