Mathematics Competition

Indiana University of Pennsylvania

2012

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not necessarily expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. If $\sqrt{x-2}+\sqrt{x+3}=5$, then x^2-6 is equal to:

   A. 6
   B. 94
   C. 30
   D. 0
   E. 138

---

2. When $\sin(\theta)\csc(-\theta)$ is defined, the expression $\sin(\theta)\csc(-\theta)$ is equivalent to:

   A. -1
   B. 1
   C. $\tan(\theta)$
   D. $\sin(1)$
   E. none of these

---

3. The center point of the sphere given by the equation $(y-2)^2+(z+5)^2+9=16-x^2-6x$ is:

   A. $(6,\ 2,\ -5)$
   B. $(3,\ -2,\ 5)$
   C. $(-3,\ 2,\ -5)$
   D. $(3,\ 2,\ -5)$
   E. none of these

---

4. Two circles have exactly one point in common. The radius of one circle is 7, while the radius of the other circle is 4. If X is a point on one circle and Y is a point on the other circle, what is the maximum possible length of the line segment XY?

   A. 11
   B. 13
   C. 14.5
   D. 18
   E. 22

---

5. If the graph of $y=\log_a x$ passes through the point $(e,\ 2)$, then a is:

   A. 1
   B. 2
   C. 1/2
   D. e
   E. none of these

---

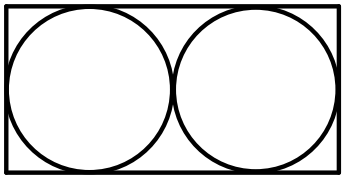
6. If the area of each circle in the figure below is 25π, then the area of the rectangle is:
   A. 125

   B. 200

   C. 200π

   D. 100

   E. none of these



---

7. The asymptotes of the function $f(x) = \dfrac{2x^2 + 5x + 7}{x^2 + 7x + 12}$ are:

   A. $x = 0$ and $y = 0$

   B. $x = -3$ and $x = -4$ and $y = 2$

   C. $x = -3$ and $x = -4$ and $y = 0$

   D. $y = 2x + 3$

   E. $f(x)$ has no asymptotes

---

8. A certain group of students is two-thirds males and one-third females. Of these, a tenth of the males are color-blind. What is the probability that a student selected at random will be a color-blind male?

   A. 1/15

   B. 1/30

   C. 1/10

   D. 1

   E. none of these

---

9. The solution set, in interval notation, of $-x^2(x-1)(x-3)^3 \geq 0$ is:

   A. $(-\infty, 0) \cup (0, 1) \cup (3, \infty)$

   B. $[1, 3]$

   C. $(-\infty, 0] \cup [1, 3]$

   D. $(-\infty, 0] \cup [3, \infty)$

   E. none of these

---

10. If $a = \frac{5}{7}b - 2$ and also $a = \frac{3-b}{4}$, then the value of a is:

A. 29/27

B. 1/27

C. 77/27

D. 29/23

E. none of these

---

11. For positive integers a and b, a lattice path in the plane from (0, 0) to (a, b) is a string consisting of N's (for North) and E's (for East) which describes a path from (0, 0) to (a, b). The number of distinct lattice paths from (0, 0) to (5, 7) is equal to:

A. 35

B. 12

C. 5^7

D. 7^5

E. none of these

---

12. The solution set of the system of inequalities

$$\begin{aligned} x - y &\leq 2 \\ x + 2y &\geq 8 \\ y - 4 &\leq 0 \end{aligned}$$

forms a polygonal region in the xy-plane. The coordinates of the vertices of this region are:

A. (4,0), (2,4), (4,6)

B. (0,2), (2,2), (4,4)

C. (0,6), (4,0), (2,4)

D. (0,4), (4,2), (6,4)

E. (0,0), (2,6), (4,2)

---

13. If $\log_{10}\left[\log_{10}\left(2 + \log_2(x+1)\right)\right] = 0$, then x is equal to:

A. 2

B. 127

C. 256

D. 255

E. none of these

---

14. A man is next to a river at the edge of a desert. It is a 6 day walk across the desert. But, he can only carry enough water for 3 days. Assume he cannot walk in the desert at all without water, he has an unlimited supply of water and containers at the river, and he can leave containers of water in the desert for future use. (For example: he can take 3 containers of water from the river, drink one during a day's walk, leave one there in the desert, then drink the third during his day's walk back.) What is the least number of days it will take him to cross the desert?

A. 6

B. 15

C. 26

D. 42

E. 80

---

15. The figure shows square $ABCD$. Arc BD with center at A and AC intersect at F. The extension of DF meets BC at G. Which relationship given below is true?

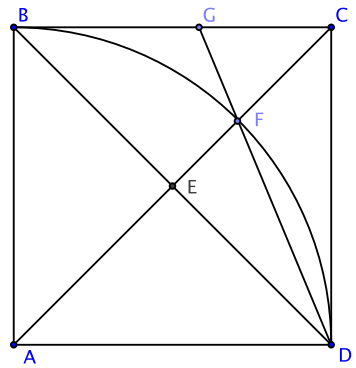
A. $BG = 2EF$

B. $BG = BC - EF$

C. $EF = FC$

D. $EF = \frac{3}{2}BG$

E. none of these



---

16. For triangle $\triangle ABC$, you are given that $\sin(B) = 1/3$, the length of AB is 1, and the length of BC is 3. The area of the triangle is:

A. $\sqrt{3}$

B. $1/2$

C. 3

D. $3\sqrt{3}$

E. 9

---

17. If you simplify the expression

$$\frac{\sqrt{b}}{\sqrt{b}+\sqrt{y}}$$

the correct result is:

A. $\dfrac{b+\sqrt{by}}{b+y}$

B. $\dfrac{b-\sqrt{by}}{b-y}$

C. $\dfrac{b-b\sqrt{y}}{b-y}$

D. $\dfrac{b+b\sqrt{y}}{b+y}$

E. $\dfrac{b-\sqrt{by}}{b+y}$

---

18. The solution set of $|2^x - 5| = 3$ is:

A. $\{\ \}$

B. $\{-1\}$

C. $\{3\}$

D. $\{1, 3\}$

E. none of these

---

19. It takes a pump 12 hours to fill an empty tank and the flood gate empties the tank in 20 hours. The total amount of time it would take the pump to fill the tank from empty if the floodgate was accidently opened when the tank is halfway full is:

A. 15 hours

B. 20 hours and 48 minutes

C. 21 hours

D. 22 hours and 45 minutes

E. 23 hours and 12 minutes

---

20. For real numbers x and y, define $x \circ y = x + y + xy$. The value of a for which $a \circ x + 5 = 0$ does not have a solution is:

A. 0

B. 1

C. -1

D. 5

E. none of these

---

21. Captain Ahab sights a whale at a distance of 300 m at an angle of 15° to the right of his direction of travel. He also sights another ship at an angle of 75° to the left of his direction of travel at a distance of 400 m from his ship. The distance between the whale and the other ship is:

A. 200 m

B. 300 m

C. 400 m

D. 500 m

E. 2500 m

---

22. The intersection of the planes $7x + 5y + 4z = 1$ and $5x + 7y + 8z = 8$ is given by the equation:

A. $y = -3x - 2$

B. $y = 3x + 2$

C. $y = -5x - 3$

D. $y = 2x + 3$

E. the planes do not intersect

---

23. How many x-intercepts could the graph of a 5th degree polynomial possibly have?

A. 0

B. 5

C. 0, 1, 2, 3, 4, or 5

D. 1, 2, 3, 4, or 5

E. 1, 3, or 5

---

24. In the given figure, Z is the center point of the circle. If $\angle A = a$ and $\angle ZBA = b$, then the measure of $\angle ZCA$ is:

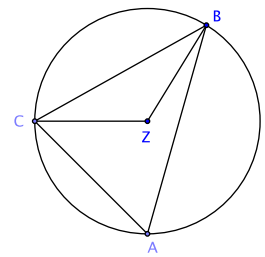
A. $a - b$

B. $a + b$

C. $180 - a + b$

D. $180 - a - b$

E. $180 - \sqrt{a^2 + b^2}$



---

25. The solution set to the system of equations

$$x^2 = 8x - 3y$$
$$y = x - 2$$

is:

A. $\{(2,\ 0),\ (3,\ 1)\}$

B. $\{(-2,\ -4),\ (-3,\ -5)\}$

C. $\{(-1,\ -3),\ (6,\ 4)\}$

D. $\{(1,\ -1),\ (-6,\ -8)\}$

E. none of these

---

26. The quotient and remainder of $(x^3 - 5) \div (x - 1)$ are:

A. $Q(x) = x^2 + x + 1$ and $R(x) = -4$

B. $Q(x) = x^3 - x^2 - x - 1$ and $R(x) = 0$

C. $Q(x) = x^2 - x - 1$ and $R(x) = 0$

D. $Q(x) = x^2 + x + 1$ and $R(x) = 5$

E. $Q(x) = x^3 + x^2 + x - 4$ and $R(x) = 0$

---

27. A *permutation* of the numbers $1, 2, \ldots, n$ is an n-tuple of elements $\sigma = (s_1,\ s_2,\ \ldots,\ s_n)$, where each s_i is a number from 1 to n and $s_i \neq s_j$ when $i \neq j$. For example, $\sigma = (5, 1, 2, 4, 3)$ is a permutation of the numbers 1 through 5. A permutation σ is called a *231-avoiding permutation* if there do not exist indices $i < j < k$ such that $s_k < s_i < s_j$. The following are 231-avoiding permutations except:

A. $\sigma = (4, 1, 3, 2)$

B. $\sigma = (2, 1, 4, 3)$

C. $\sigma = (3, 2, 1, 4)$

D. $\sigma = (3, 4, 1, 2)$

E. all are 231-avoiding

---

28. A ping-pong ball of radius 1 and a bowling ball are in an empty cubical room. The bowling ball is trying to crush the ping-pong ball so the ping-pong ball hides in the corner. If the bowling ball can not reach the ping-pong ball, the radius of the big bowling ball must be greater than:

A. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$

B. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

C. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

D. $\dfrac{\sqrt{3}}{2}+1$

E. $3+2\sqrt{2}$

---

29. Suppose x, y, and z are all integers and that $x^2 = yz$. If we know that y is even but not divisible by 4, then a possible value of z is:

A. 100

B. 40

C. 12

D. 25

E. none of these

---

30. In triangle $\triangle ABC$, angles A and B have the same measure, while the measure of angle C is $90°$ more than each of A and B. The measures of the three angles are:

A. $A = 120°$, $B = 120°$, and $C = 30°$

B. $A = 30°$, $B = 30°$, and $C = 120°$

C. $A = 40°$, $B = 40°$, and $C = 120°$

D. $A = 100°$, $B = 100°$, and $C = 40°$

E. none of these

---

31. The solution set of the equation $\dfrac{4}{x^2+2x+4} = \dfrac{x}{x-2} - \dfrac{8}{x^3-8}$ is:

A. $\{0,\ 2,\ -2\}$

B. $\{-2\}$

C. $\{0,\ 2\}$

D. $\{2,\ -2\}$

E. $\{0,\ -2\}$

---

32. When A and B represent logical statements, the symbol $\vee$ is the logical *or*, the symbol $\wedge$ is the logical *and*, and the symbol $\neg$ is the logical *negation*. The logical statement $A \vee (\neg A \wedge B)$ is:

A. always true

B. equivalent to $A \wedge \neg B$

C. equivalent to $A \vee B$

D. equivalent to $A \wedge B$

E. equivalent to $A \vee \neg B$

---

33. Given that $\angle x = (6n - 18)^\circ$, $\angle y = (4n - 11)^\circ$, and $\angle z = (169 - n)^\circ$ in the triangle below, the measure of $\angle z$ is:

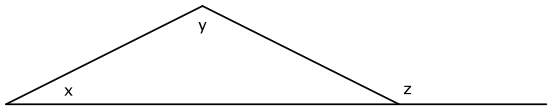
A. 153°

B. 73°

C. 18°

D. 161°

E. none of these



---

34. The solution set to the equation $4^{x+2} - 2^{x+4} = 2^x - 1$ is:

A. $\{\ \}$

B. $\{0\}$

C. $\{1\}$

D. $\{-3, 1\}$

E. none of these

---

35. The expression

$$(\sin\beta + \cos\beta)^2 - A = B \sin\beta \cos\beta$$

is an identity when the constants A and B are:

A. $A = 1, B = 2$

B. $A = 1, B = -2$

C. $A = 2, B = 1$

D. $A = 1, B = 0$

E. none of these

---

36. If $x < -2$, then $\dfrac{|2x+4|}{|x|+2}$ is equal to:

A. 2

B. -2

C. $\dfrac{2(x+2)}{x-2}$

D. $\dfrac{2x+4}{2-x}$

E. none of these

---

37. Given that $\angle A = (7x-57)^\circ$ and $\angle B = (5x-25)^\circ$ as shown below, the values for $\angle A$ and $\angle B$ are:

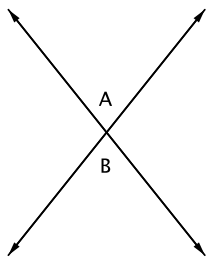
A. 125° and 125°

B. 55° and 55°

C. 16° and 16°

D. 125° and 55°

E. none of these



---

38. The expression $\log_2\left(\left(\sqrt[3]{2}\right)^{\frac{1}{5}}\right)$ simplifies to:

A. $\frac{1}{10}$

B. $\frac{3}{5}$

C. $\frac{1}{15}$

D. $\frac{3}{10}$

E. $\frac{5}{3}$

---

39. Let f be a function whose domain is the set X and whose range is a subset of Y. Such a function is called *injective* whenever $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for any x_1 and x_2 in set X. If X has 5 elements and Y has 8 elements, then the number of possible injective functions is equal to:

A. 13

B. 40

C. 120

D. 6720

E. none of these

---

40. The figure shows square $ABCD$ where E, F, G, and H are midpoints of the sides. If $AB = a$, the length of MN is:

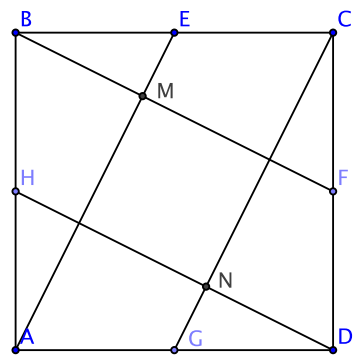
A. $\frac{a\sqrt{5}}{5}$

B. $\frac{a\sqrt{10}}{5}$

C. $\frac{a\sqrt{5}}{2}$

D. $\frac{\sqrt{10}}{a}$

E. none of these



---

41. The *lead* of a screw is the distance that the screw advances in a straight line when the screw is turned 1 complete turn. If a screw is $2\frac{1}{2}$ inches long and has a lead of an eighth of an inch, the number of complete turns it will take to get the screw all the way into a piece of wood is:

A. 5

B. 10

C. 15

D. 20

E. 25

---

42. If $\log_{\frac{1}{2}}(-2v) = 4$ then v is equal to:

A. -4

B. $-\frac{1}{32}$

C. -1

D. $-\frac{1}{8}$

E. -16

---

43. Find all real numbers where the cube root of the number is 6 more than the principal sixth root of the number. The sum of all such numbers is:

A. 1243

B. 729

C. 793

D. 243

E. 144

---

44. A small town is separated from the local power plant by mountainous terrain and several lakes. Until now, electrical power has been routed through a nearby city. The recent development of a stronger wire permits a direct line to be constructed. Sighting from the town, the angle between the city and the power plant is 30°. The distance between the city and the town is 200 km. The distance between the power plant and the city is 100 km. The straight line distance from the town to the power plant is:

A. $106\sqrt{3}$ km

B. $10\sqrt{106}$ km

C. 100 km

D. $50\sqrt{2}$ km

E. $100\sqrt{3}$ km

---

45. A square with side lengths of 3 inches is inscribed inside of a circle. Find the area of the region that is both inside of the circle and outside of the square.

A. $3\pi - 3$

B. $9\pi - 9$

C. $\frac{9\pi}{2} - 9$

D. $\frac{3\pi}{2} - 9$

E. $18\pi - 9$

---

46. The supplement of an angle measures 66° less than 4 times its complement. The measure of the angle is:

A. 38°

B. 151°

C. 75.5°

D. 19°

E. none of these

---

47. The sum of all of the solutions of the equation $x^3 + 2x^2 - 12x - 9 = 0$ is:

A. 3

B. 0

C. -7

D. -2

E. none of the above

---

48. For a non-negative integer n, a binary tree having n vertices is defined recursively as follows:

    1. The empty set, $\emptyset$, denotes the unique binary tree having 0 vertices.
    2. If T_1 and T_2 are binary trees having m and n vertices respectively, then $T = (\bullet, T_1, T_2)$ is a binary tree having 1+m+n vertices.

    For example, $T_e = (\bullet, (\bullet, \emptyset, (\bullet, \emptyset, \emptyset)), \emptyset)$.

    The correct representation for the tree T_p is:

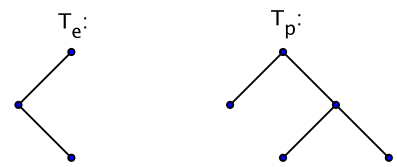
    A. $T_p = (\bullet, \bullet, \bullet, \bullet, \bullet, \emptyset)$

    B. $T_p = (\bullet, (\bullet, \emptyset, \emptyset), (\bullet, (\bullet, \emptyset, \emptyset), (\bullet, \emptyset, \emptyset)))$

    C. $T_p = (\bullet, (\bullet, \emptyset, \emptyset), (\bullet, (\bullet, (\bullet, \emptyset, \emptyset), \emptyset), \emptyset))$

    D. $T_p = (\bullet, (\bullet, (\bullet, \emptyset, \emptyset), \emptyset), (\bullet, (\bullet, \emptyset, \emptyset), \emptyset))$

    E. none of these



---

49. The solution set, in interval notation, of the inequality $|13x - 7| < -2$ is:

    A. $\left(\frac{5}{13}, \frac{9}{13}\right)$
    B. $\left(-\infty, \frac{5}{13}\right) \cup \left(\frac{9}{13}, \infty\right)$
    C. $(-\infty, \infty)$
    D. $\left(-\infty, \frac{5}{13}\right)$
    E. No solution

---

50. Let

$$\begin{aligned} ab + 2ad &= 5\sqrt{5} \\ 2a - (b + 2d)c &= 0 \\ 2b - ac &= 0 \\ 2d - 2ac &= 0 \ . \end{aligned}$$

    For this system of equations, one value of $a + b + c + d$ is:

    A. $\dfrac{15 + 7\sqrt{5}}{5}$

    B. $\dfrac{15 - 7\sqrt{5}}{5}$

    C. $\dfrac{-3 + 7\sqrt{5}}{5}$

    D. 0

    E. $\dfrac{3 + 15\sqrt{5}}{5}$

Answer Key

1. C
2. A
3. C
4. E
5. E
6. B
7. B
8. A
9. E
10. B
11. E
12. D
13. D
14. D
15. A
16. B
17. B
18. D
19. C
20. C
21. D
22. A
23. D
24. A
25. C
26. A
27. D
28. B
29. B
30. B
31. E
32. C
33. E
34. E
35. A
36. C
37. B
38. C
39. D
40. B
41. D
42. B
43. B
44. E
45. C
46. A
47. D
48. B
49. E
50. A